Communication constraints in coordinated consensus problems

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***Abstract*— The coordinated consensus problem can be seen as one of the simplest instance of coordinated control. This problem has been widely investigated in the recent years. It is clear that the information exchange must have an important influence on the performance of the control strategy. In this contribution the information flow is modelled by a graph representing the information transmission from one vehicle to another one. In this graph there are two kinds of edges. One kind of edges represents exact data transmission. This is very expensive with respect to communication rate required. A second kind of edges represents transmission logarithmic quantized data. On the contrary this is very cheap with respect to communication rate required. The final goal of the present paper is to determine how the degree of connection of this graph influences the performance of the coordinated control system.**

I. INTRODUCTION

Coordination algorithms for multiple autonomous vehicles have attracted large attention in recent years. This is mainly motivated by that multi-vehicle systems have application in many areas, such as coordinated flocking of mobile vehicles [14], [15], cooperative control of unmanned air and underwater vehicles [18], [17], attitude alignment of clusters of satellites [16].

In many applications, coordinating vehicles need to communicate data in order to execute a task. In particular they may need to agree on the value of certain state variables. Such problem, called *coordinated consensus* problem, has attracted a great attention in the control community, see for example [1], [2], [3], [4] and reference therein.

It is clear that control performance has to be strictly related to the amount of information exchange among vehicles. More precisely, if we model the information flow by a graph representing the information transmission from one vehicle to another one, we can expect that good control design methods have to yield better performance for graphs that are more connected.

The situation mostly treated in the literature is when each vehicle has the possibility of communicate its state to the other vehicles that are positioned inside a neighborhood. In this case the graph changes over time and a precise analysis of how the communication constraints influence the performance is rather difficult. A situation that is simpler and easier to be analyzed is the one in which the graph is fixed and the controller is time-invariant. These assumptions could be considered too restrictive in most of the applications. Nevertheless, this is one of the simplest model one can consider and a clear analysis seems a necessary preliminary step before passing to more realistic models.

The present contribution analyzes different situations. First we relate the consensus stability to the structure of the graph representing the communication topology. This analysis is similar to what proposed in [2]. However, here there is much more emphasis on the relationships between the graph properties and the control performance.

Another important contribution of the present paper is to relate control under communication constraints and coordinated control. This is done by introducing, in the graph describing the data exchange, another kind of edge that represents the transmission of logarithmic quantized data. Although exact data transmission is very expensive with respect to the required communication rate, it is well-known [5] that this second way of transmitting data is, on the contrary, very cheap. A preliminary analysis of coordinated control strategies involving logarithmic quantized data transmission has been proposed in [6]. This analysis is very complicated in general. For this reason here we restrict our attention to graph respecting a symmetry. This approach is very much on the lines proposed in [4]. In this case a precise analysis is possible. Finally, through some examples, it is shown that logarithmic quantized data transmission can improve the performance and so they can be considered as new ingredient for improving coordination control design strategies.

II. PROBLEM FORMULATION

Consider $N > 1$ identical systems whose dynamics are described by the following discrete time state equations

$$x_i^+ = x_i + u_i \qquad i = 1, \ldots, N$$

where $x_i \in \mathbb{R}$ is the state of the i-th system, x_i^+ represents the updated state and $u_i \in \mathbb{R}$ is the control input. More compactly we can write

$$x^+ = x + u$$

where $x, u \in \mathbb{R}^N$. The goal is to design a feedback control

$$u = Kx, \qquad K \in \mathbb{R}^{N \times N}$$

yielding the consensus of the states, namely a control such that all the x_i's become equal asymptotically. More precisely, our objective is to obtain K such that, for any initial condition $x(0) \in \mathbb{R}^N$, the closed loop system

$$x^+ = (I + K)x \, ,$$

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yields

$$\lim_{t\to\infty} x(t) = \alpha v \tag{1}$$

where $v := (1, \dots, 1)^T$ and where α is a scalar depending on $x(0)$.

The fact that in the matrix K the element in position i, j is different from zero, means that the systems i needs to know the state of the systems j in order to compute its feedback action. This implies that we need to communicate the state x_j from the system j to the system i. A good description of the communication effort required by a specific feedback K is given by the directed graph $\mathcal{G}_K$ with set of vertices $\{1, \dots, N\}$ in which there is an arc from j to i whenever in the feedback matrix K the element $K_{ij} \neq 0$. The graph $\mathcal{G}_K$ is said to be the *communication graph* associated with K. Conversely, given any directed graph $\mathcal{G}$ with set of vertices $\{1, \dots, N\}$, we say that a feedback K is *compatible* with $\mathcal{G}$ if $\mathcal{G}_K$ is a subgraph of $\mathcal{G}$ (we will use the notation $\mathcal{G}_K \subseteq \mathcal{G}$). We say that the consensus problem is solvable on a graph $\mathcal{G}$ if there exists a feedback K compatible with $\mathcal{G}$ and solving the consensus problem. From now on we always assume that $\mathcal{G}$ contains all loops (i, i) meaning that each system has access to its own state.

The problem we are considering is that of obtaining a feedback matrix K, compatible with a given graph, that yields the consensus and minimizes a suitable performance index. We would like to relate the connectivity of $\mathcal{G}_K$ to the achievable performance. More precisely suppose we have chosen a cost functional $\mathcal{R}(K)$ that describes the performance and that we want to minimize. Then we can then define

$$\mathcal{R}_{\mathcal{G}} = \min\{\mathcal{R}(K) \mid \mathcal{G}_K \subseteq \mathcal{G}\}\,.$$

A meaningful cost functional should be sensitive to the communication effort. In other words we expect that a good choice of $\mathcal{R}(K)$ will produce an index $\mathcal{R}_{\mathcal{G}}$ that shows a certain range of variation among all the possible communication graphs that can be considered.

One of the simplest control performance index is the exponential speed of convergence to the consensus. However, such index does not exhibit the desired sensitivity to the communication effort and it would yield completely meaningless optimal solutions. Indeed, simply by choosing $K = -I$, we would obtain the best possible performance with zero communication effort. However this solution is clearly not effective since, in this way, nonzero initial states having equal components, and hence in which we have already consensus, would produce a useless control action driving all the states to zero.

These kind of solutions are automatically discarded if we restrict to feedback matrices K ensuring that the set of equilibrium points of the closed loop system coincides with the subspace generated by the vector v. This happens if and only if

$$Kv = 0\,. \tag{2}$$

¿From now on we impose this condition on K. It is easy to see that the consensus problem is solved if and only if the following three conditions hold:

(A) the only eigenvalue of $I + K$ on the unit circle is 1;
(B) the eigenvalue 1 has algebraic multiplicity one (namely it is a simple root of the characteristic polynomial of $I + K$) and v is its eigenvector;
(C) all the other eigenvalues are strictly inside the unit circle.

Under these conditions the speed of convergence can be defined as follows. Let P be any matrix such that $Pv = v$ and assume that its spectrum $\sigma(P)$ is contained in the closed unit disk centered in 0. Define

$$\rho(P) = \begin{cases} 1 & \text{if } \dim\ker(P - I) > 1 \\ \max_{\lambda\in\sigma(P)\setminus\{1\}} |\lambda| & \text{if } \dim\ker(P - I) = 1\,, \end{cases} \tag{3}$$

Even though the consensus under condition (2) requires information exchange between the systems, this condition is not enough to make $\rho(I + K)$ a reasonable performance index. Indeed, if we take the graph $\mathcal{G}$ described by $1 \to 2 \to \cdots \to N$, we have that using the controller

$$K = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

we achieve the maximum possible performance. So, in this case, adding communication edges would yield no improvement and this is not a reasonable conclusion.

We need therefore to refine the model. The weakness of the previous model is caused by that the input is inexpensive. A better performance index could be obtained by adding to the rate of convergence a cost function depending on the input, such as a quadratic cost like

$$J(K) := \mathbb{E}\left(\sum_{t=0}^{\infty} ||u(t)||^2\right)$$

where the expected value is calculated with respect to a probability distribution of the initial state $x(0)$. The performance index consisting of the pair $(\rho(I+K), J(K))$ would yield a much better description of the problem. However the consequent optimization problem would be non convex and so also this solution is unacceptable.

Therefore we consider a simpler index that is related to the previous one, namely

$$J'(K) := \mathbb{E}\left|\left|\sum_{t=0}^{\infty} u(t)\right|\right|^2\,.$$

Notice that it is clearly a lower bound of $J(K)$. In our case, we have that

$$\begin{aligned} J'(K) &= \mathbb{E}||x(\infty) - x(0)||^2 \\ &= N^2\mathbb{E}\left(\alpha(x(0)) - \frac{1}{N}v^T x(0)\right)^2 \\ &\quad + \mathbb{E}\left|\left|\frac{1}{N}(v^T x(0))v - x(0)\right|\right|^2 \end{aligned}$$

where $\alpha(x(0))$ is the scalar satisfying (1). Notice that the second term of the previous addition is independent of K and so the previous cost function is equivalent to

$$J''(K) := \mathbb{E}\left(\alpha(x(0)) - \frac{1}{N}v^T x(0)\right)^2 ,$$

which coincides with the average distance of the consensus point from the barycenter of the initial states. In this paper we propose a control performance index consisting of the pair $(\rho(I+K), J''(K))$, which is significant and also treatable. In fact it is easy to see that $J''(K) = 0$ if and only if

$$v^T K = 0 \tag{4}$$

These feedback maps are called consensus controllers [2]. When K yields such a behavior, it will be called a *barycentric controller*. Notice that this condition is automatically true for symmetric matrices K satisfying (2). From this choice of performance index we can formulate the following control problem.

Problem: Given a graph $\mathcal{G}$, find a matrix K satisfying (2) and (4) such that $\mathcal{G}_K \subseteq \mathcal{G}$ and minimizing $\rho(K)$.

When we are dealing with barycentric controllers it is meaningful to consider the displacement from the barycenter

$$\Delta(t) := x(t) - \left(\frac{1}{N}v^T x(0)\right) v\,.$$

It is immediate to check that, $\Delta(t) = x(t) - \left(\frac{1}{N}v^T x(t)\right) v$ and that it satisfies the closed loop equation

$$\Delta^+ = (I+K)\Delta\,. \tag{5}$$

Notice moreover that the initial conditions $\Delta(0)$ are such that

$$v^T \Delta(0) = 0\,. \tag{6}$$

Hence the asymptotic behavior of our consensus problem can equivalently be studied by looking at the evolution (5) on the hyperplane characterized by the condition 6. The index $\rho(I+K)$ seems in this context quite appropriate for analyzing how performance is related to the communication effort associated with a graph.

III. PROBLEM SOLUTION: STOCHASTIC AND DOUBLY STOCHASTIC MATRICES

If we restrict to control laws K making $I+K$ a nonnegative matrix, namely a matrix with all nonnegative entries, condition (2) imposes that $I+K$ is a stochastic matrix and $\rho(I+K)$ is called the *spectral radius*. Since the spectral structure of stochastic matrices is quite well known, this observation allows to understand easily what conditions on the graph ensure the solvability of the consensus problem. To exploit this we need to recall some notation and results on directed graphs (the reader can further refer to textbooks on graph theory such as [12] or [10]).

Fix a directed graph $\mathcal{G}$ with set of vertices V and set of arcs $\mathcal{E} \subseteq V \times V$. The adjacency matrix A is a $\{0,1\}$-valued square matrix indexed by the elements in V defined by letting $A_{ij} = 1$ if and only $(i,j) \in \mathcal{E}$. Define the in-degree of a vertex j as $\mathrm{indeg}(j) := \sum_i A_{ij}$ and the out-degree of a vertex i as $\mathrm{outdeg}(i) := \sum_j A_{ij}$. Vertices with out-degree equal to 0 are called sinks. A graph is called in-regular (out-regular) of degree k if each vertex has in-degree (out-degree) equal to k. A path in $\mathcal{G}$ consists of a sequences of vertices $i_1 i_2 \ldots\ldots i_r$ such that $(i_\ell, i_{\ell+1}) \in \mathcal{E}$ for every $\ell = 1, \ldots, r-1$; i_1 (resp. i_r) is said to be the initial (resp. terminal) vertex of the path. A cycle is path in which the initial and the terminal vertices coincide. A vertex i is said to be connected to a vertex j if there exists a path with initial vertex i and terminal vertex j. A directed graph is said to be connected if, given any pair of vertices i and j, either i is connected to j or j is connected to i. A directed graph is said to be strongly connected if, given any pair of vertices i and j, i is connected to j.

Given any directed graph $\mathcal{G}$ we can consider its strongly connected components, namely strongly connected subgraphs $\mathcal{G}_k$, $k = 1, \ldots, s$, with set of vertices $V_k \subseteq V$ and set of arcs $\mathcal{E}_k = \mathcal{E} \cap (V_k \times V_k)$ such that the sets V_k form a partition of V. The various components may have connections among each other. We define another directed graph $T_{\mathcal{G}}$ with set of vertices $\{1, \ldots, s\}$ such that there is an arc from h to k if there is an arc in $\mathcal{G}$ from a vertex in V_h to a vertex in V_k. It can easily be shown that $T_{\mathcal{G}}$ is a graph without cycles. The following proposition is the straightforward consequence of a standard results on stochastic matrices [11, pag. 88 and pag. 95].

Proposition 3.1: Let $\mathcal{G}$ be a directed graph and assume that $\mathcal{G}$ contains all loops (i,i). The following conditions are equivalent:

(i) The consensus problem is solvable on $\mathcal{G}$.
(ii) $T_{\mathcal{G}}$ is connected and has only one sink vertex.

Moreover, if the above conditions are satisfied, any K such that $I+K$ is stochastic, $\mathcal{G}_K = \mathcal{G}$ and $K_{ii} \neq -1$ for every $i = 1, \ldots, n$ is a solution of the consensus problem.

Among all possible solutions of the consensus problem, for which the graph $\mathcal{G}$ satisfies the properties of Proposition 3.1, a particularly simple one can be defined. Let P be a matrix defined as follows[1]

$$P_{ij} = \begin{cases} \dfrac{1}{\mathrm{indeg}(i)} & \text{if } i \to j \\[2ex] 0 & \text{otherwise} \end{cases}.$$

Then the matrix $K := P - I$ solves the consensus problem. Notice that the matrix P defined above is called the Laplacian of the graph. In this case the closed loop dynamics have the following form

$$x_i^+ = x_i + \frac{1}{\mathrm{indeg}(i)} \sum_{\substack{j \neq i \\ (j,i) \in \mathcal{E}}} (x_j - x_i)\,. \tag{7}$$

[1]In the following we use the notation $i \to j$ to indicate that there is an arc from vertex i to vertex j. With the notation $i \leftarrow j$ we indicate the viceversa.

If we restrict now to control laws K making $I + K$ a nonnegative matrix, conditions (2)) and (4) are equivalent to imposing that $I + K$ is a doubly stochastic matrix. The following proposition is again a straightforward consequence of a standard results on stochastic matrices [11, pag. 88 and pag. 95].

Proposition 3.2: Let $\mathcal{G}$ be a directed graph and assume that $\mathcal{G}$ contains all loops (i,i). The following conditions are equivalent:

(i) The barycentric consensus problem is solvable on $\mathcal{G}$.
(ii) $\mathcal{G}$ is strongly connected.

Moreover, if the above conditions are satisfied, any K such that $I + K$ is doubly stochastic, $\mathcal{G}_K = \mathcal{G}$ and $K_{ii} \neq -1$ for every $i = 1, \ldots, n$ is a possible solution.
Notice that in the special case when the graph $\mathcal{G}$ is undirected, namely $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$, we can find matrices K, that solve the consensus problem, that are symmetric and stochastic, and therefore automatically doubly stochastic. One example is given by (7).

We expect the spectral radius to be a meaningful cost functional when restricted to feedback controllers K such that $I+K$ is doubly stochastic. More precisely we conjecture that, by taking

$$\rho_{\mathcal{G}}^{\mathrm{ds}} = \min\{\rho(K) \mid I + K \text{ doubly stochastic}, \, \mathcal{G}_K \subseteq \mathcal{G}\}\,,$$

$\mathcal{G}_1 \subset \mathcal{G}_2$ implies that $\rho_{\mathcal{G}_1} > \rho_{\mathcal{G}_2}$. However we have been not able to prove this so far. The problem of minimizing $\rho(K)$ or, equivalently, of maximizing $1 - \rho(K)$ (which is called the spectral gap of the associated Markov chain) is a very classical problem in the theory Markov chains. Recently some very effective algorithms have been proposed for this maximization limited to the case in which K is a symmetric matrix.

IV. SYMMETRIC CONTROLLERS

The analysis of the rendezvous problem and the corresponding controller synthesis problem becomes more treatable if we limit our considerations to graphs $\mathcal{G}$ and matrices K with symmetries. We here limit ourselves to the cyclic symmetry. However other possibilities can be considered [4]. Let

$$p : \{1, \ldots, N\} \to \{1, \ldots, N\} : i \mapsto i + 1 \bmod N$$

The feedback matrix K is symmetric with respect to p if

$$K_{i,j} = K_{p(i),p(j)}, \quad \forall i, j \in \{1, \ldots, N\}\,.$$

This condition is equivalent to impose that

$$K = \sum_{i=0}^{N} k_i \Pi^i$$

where

$$\Pi := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

These matrices are called circulant [7]. Notice that, in this case, since $\Pi v = v$ and $v^T \Pi = v^T$, then circulant feedback matrices satisfy condition (2) if and only if $\sum_i k_i = 0$. Moreover condition (4) is also satisfied, and thus such controllers drive the state to the barycenter. Consequently, if we choose K such that $I + K$ is nonnegative, then $I + K$ is always doubly stochastic. The spectral properties of circulant matrices are particularly simple. Indeed, it can be shown that

$$\sigma(I + K) = \{1 + K(e^{j\frac{2\pi}{N}h}) : h = 0, 1, \ldots, N - 1\}$$

where $K(z) := \sum_{i=0}^{N} k_i z^i$. Hence,

$$\rho(K) = \max\{|1 + K(e^{j\frac{2\pi}{N}h})| : h = 1, \ldots, N - 1\}\,.$$

Moreover, the corresponding eigenvectors v_h's form an orthonormal basis and $v_0 = (1/N)v$.

Notice that, in order to have rendezvous stability in this context, it is sufficient to impose that

$$|1 + K(e^{j\theta})| < 1 \quad \forall \theta \neq 0\,. \tag{8}$$

This condition is slightly stronger than rendezvous stability, however it provides a stability condition independent of the number of vehicles N.

Circulant solutions to the rendezvous problem exist if the graph $\mathcal{G}$ admits an analogous symmetry. Indeed, consider a strongly connected graph $\mathcal{G}$ on $\{1, \ldots, N\}$ that is symmetric with respect to p, in the sense that if there is an arc from i to j, there is also an arc from $p(i)$ to $p(j)$. Then, it is immediate to find a circulant matrix K such that $\mathcal{G}_K = \mathcal{G}$ and that solves the rendezvous problem. Indeed if $j_1, \ldots, j_\mu$ are the incoming arcs of vertex 1 in $\mathcal{G}$, it is sufficient to choose weights $k_0, \ldots, k_\mu$ such that $k_i > 0$ for $i = 1, \ldots, \mu$, $k_0 > -1$ and $\sum_i k_i = 0$. If we consider $K = \sum k_i \Pi^i$, then it is clear that condition (8) is satisfied.

In the context of circulant matrices, the spectral radius is a reasonably simple cost functional. Define

$$\rho_{\mathcal{G}}^{\mathrm{circ}} = \inf\{\rho(K) \mid K \text{ circulant}\,,\, \mathcal{G}_K \subseteq \mathcal{G}, I+K \text{ stochastic}\}\,.$$

Example 1: Suppose $\mathcal{G}$ is described by the arcs $i \leftarrow i + 1$ (mod N). We can choose therefore $K(z) = k_0 + k_1 z$, where $k_0, k_1 \in \mathbb{R}$. In this case we have that

$$x^+ = \{I + k_0 I + k_1 \Pi\} x\,.$$

The condition $K(1) = k_0 + k_1 = 0$ implies that $K(z) = k(1 - z)$ for some $k \in \mathbb{R}$. In this case it can be shown that we have rendezvous stability if and only if $-1 < k < 0$ and that the rate of convergence is

$$\rho(K) = \left((k + 1)^2 + k^2 - 2k(k + 1)\cos\left(\frac{2\pi}{N}\right)\right)^{\frac{1}{2}}.$$

The k that minimizes $\rho(K)$ is $k = -1/2$ and yields

$$\rho_{\mathcal{G}}^{\mathrm{circ}} = \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{N}\right)\right)^{\frac{1}{2}} \simeq 1 - \frac{\pi^2}{2}\frac{1}{N^2} \tag{9}$$

where the last approximation is meant for $N \to \infty$.

Example 2: Suppose $\mathcal{G}$ is described by the arcs $i \leftarrow i - 1$ and $i \leftarrow i + 1$ (mod. N). For the sake of simplicity we assume that N is even; very similar results can be obtained when N is odd. We can choose in this case $K(z) = k_0 + k_1 z + k_{-1} z^{-1}$, where $k_0, k_1, k_{-1} \in \mathbb{R}$. Then we have that

$$x^+ = \{I + k_0 I + k_1 \Pi + k_{-1} \Pi^{-1}\} x \,.$$

The condition $K(1) = 0$ becomes in this case $k_0 + k_1 + k_{-1} = 0$. Symmetry and convexity arguments allow to say that a minimum of $\rho(K)$ is for sure of the type $k_1 = k_{-1}$. With this assumption the cost functional reduces to

$$\rho(K) = \max\{|1 - 2k_1(1 - \cos\left(\frac{2\pi}{N}\right))|, |1 + k_0 - 2k_1|\} \,.$$

The minimum is achieved for

$$k_0 = -\frac{2}{3 - \cos\left(\frac{2\pi}{N}\right)}\,, \quad k_1 = k_{-1} = \frac{1}{3 - \cos\left(\frac{2\pi}{N}\right)}$$

and we have

$$\rho_{\mathcal{G}}^{\mathrm{circ}} = \frac{1 + \cos\left(\frac{2\pi}{N}\right)}{3 - \cos\left(\frac{2\pi}{N}\right)} \simeq 1 - 2\pi^2 \frac{1}{N^2}\,, \tag{10}$$

where the last approximation is meant for $N \to \infty$.

Notice the asymptotic behavior of previous two examples: the case of communication exchange with two neighbors offer a better performance. However, in both cases $\rho_{\mathcal{G}}^{\mathrm{circ}} \to 1$ for $N \to +\infty$. This fact is more general: if we keep bounded the number of incoming edges in a vertex, the spectral radius will always converge to 1. This very easy to see in the case when there is only one incoming edge. Indeed in this case simply repeating the arguments of Example 1 we have the following result.

Proposition 4.1: Consider a strongly connected graph $\mathcal{G}$ on $\{1, \dots, N\}$ which is symmetric with respect to p and assume there is only one incoming edge in any vertex. Then,

$$\rho_{\mathcal{G}}^{\mathrm{circ}} \geq 1 - \frac{\pi^2}{2} \frac{1}{N^2}\,.$$

In the general situation a much more careful analysis, carried out in [13], permits to obtain the following bound.

Theorem 4.1: Consider a strongly connected graph $\mathcal{G}$ on $\{1, \dots, N\}$ that is symmetric with respect to p and let ν be the number of incoming edges in any vertex. Then,

$$\rho_{\mathcal{G}}^{\mathrm{circ}} \geq 1 - C \frac{1}{N^{2/\nu}}\,, \tag{11}$$

where C is a constant independent of the chosen graph.

Now we can wonder whether it is possible to achieve the bound performance. In other words, is the lower bound we have just found, tight? In the following example we will show that this is the case.

Example 3: Suppose That $N = M^\nu$ and that

$$K = \frac{1}{\nu + 1} \sum_{i=0}^{\nu - 1} \Pi^{M^i}$$

In this way the closed loop matrix has eigenvalues

$$\lambda_h = p_\nu(e^{j\frac{2\pi}{N}h}) \qquad h = 1, \dots, N-1$$

where

$$p_\nu(z) := \frac{1}{\nu + 1}\left(1 + \sum_{i=0}^{\nu-1} z^{M^i}\right)$$

We will show that, for all $h = 1, \dots, N-1$ we have that

$$|p_\nu(e^{j\frac{2\pi}{M^\nu}h})| \leq 1 - \frac{1}{\nu+1}\frac{1}{M^2}$$

This fact will be shown by induction on ν. The fact that the assertion holds for $\nu = 1$ follows from example 1. Assume now that the assertion holds for $\nu - 1$. Let h_0, h_1 such that $0 \leq h_0 \leq M - 1$, $0 \leq h_1 \leq M^{\nu-1} - 1$ and $h = h_0 + M h_1$. If $h_0 \neq 0$ then

$$\begin{aligned}
|p_\nu(e^{j\frac{2\pi}{N}h})| &\leq \frac{1}{\nu+1}\left|1 + e^{j\frac{2\pi}{M^\nu}M^{\nu-1}h}\right| \\
&\quad + \frac{1}{\nu+1}\left|\sum_{i=0}^{\nu-2} e^{j\frac{2\pi}{M^\nu}M^i h}\right| \\
&\leq \frac{1}{\nu+1}\left|1 + e^{j\frac{2\pi}{M}h_0}\right| + \frac{\nu-1}{\nu+1} \\
&= \frac{2}{\nu+1}\left|p_1(e^{j\frac{2\pi}{M}h})\right| + \frac{\nu-1}{\nu+1} \\
&\leq \frac{2}{\nu+1}\left(1 - \frac{1}{2}\frac{1}{M^2}\right) + \frac{\nu-1}{\nu+1} \\
&\leq 1 - \frac{1}{\nu+1}\frac{1}{M^2}
\end{aligned}$$

If $h_0 = 0$, then $h = M h_1$ and so

$$\begin{aligned}
|p_\nu(e^{j\frac{2\pi}{N}h})| &= \frac{1}{\nu+1}\left|1 + \sum_{i=0}^{\nu-1} e^{j\frac{2\pi}{M^{\nu-1}}M^i h_1}\right| \\
&= \frac{1}{\nu+1}\left|2 + \sum_{i=0}^{\nu-2} e^{j\frac{2\pi}{M^{\nu-1}}M^i h_1}\right| \\
&= \frac{\nu}{\nu+1}\left|p_{\nu-1}(e^{j\frac{2\pi}{M}h_1})\right| + \frac{1}{\nu+1} \\
&\leq \frac{\nu}{\nu+1}\left(1 - \frac{1}{\nu}\frac{1}{M^2}\right) + \frac{1}{\nu+1} \\
&\leq 1 - \frac{1}{\nu+1}\frac{1}{M^2}
\end{aligned}$$

This bound proves that there exists a circulant graph $\mathcal{G}$ with ν incoming edges in any vertex such that

$$\rho_{\mathcal{G}}^{\mathrm{circ}} \leq 1 - \frac{1}{\nu+1}\frac{1}{N^{2/\nu}}\,.$$

proving in this way that the bound proposed by the previous theorem is tight.

A. Logarithmic quantizers

We want to analyze now what happens if we allow data exchange which are corrupted by multiplicative noise. This models communication links over which logarithmic quantized data are exchanged [19]. Assume we have fixed $\Lambda \subseteq \{0, \dots, N-1\}$ and assume that for all i the vehicle i knows the states x_j for all $j \in i + \Lambda$. Assume that $0 \in \Lambda$. From this knowledge, the vehicle i can produce the outputs

$$y_{s,i} = \sum_{j \in \Lambda} \bar{H}_{s,j} x_{i+j} \quad s = 1, \dots, g\,, \qquad g \in \mathbb{N}$$

(from now on the sum of indices is meant mod N) that, corrupted by the multiplicative noise $1 + e_{s,i}$, is sent to the other vehicles. The control action at each vehicle is thus

$$u_i = \sum_{j\in\Lambda} K_{0,j} x_{i+j} + \sum_{s=1}^{g}\sum_{j=1}^{N} \bar{K}_{s,j} y_{s,i+j}(1+e_{s,i+j})$$

We obtain in this way that

$$\begin{aligned} x_i^+ &= x_i + \sum_{j\in\Lambda} K_{0,j} x_{i+j} + \\ &\quad + \sum_{s=1}^{g}\sum_{j=1}^{N}(1+e_{s,i+j})\sum_{r\in\Lambda} \bar{K}_{s,j}\bar{H}_{s,r} x_{i+j+r} \end{aligned}$$

Defining the polynomials $K_0(z) := \sum_{j\in\Lambda} K_{0,j} z^j$, $\bar{K}_s(z) := \sum \bar{K}_{s,j} z^j$, $\bar{H}_s(z) := \sum \bar{H}_{s,r} z^r$ and the matrices $E_s := \mathrm{diag}\{e_{s,1},\dots,e_{s,N}\}$ the previous equations can be written in the following vector form

$$x^+ = \{I + K_0(\Pi) + \sum_{s=1}^{g} \bar{K}_s(\Pi)(I+E_s)\bar{H}_s(\Pi)\}x$$

We need to impose here that, if $x = v$, then $x^+ = v$ for all noises $E_1,\dots,E_g$. It can be shown that this happens if and only if $K_0(1) = 0$ and $\bar{H}_s(1) = 0$. This implies that

$$K_0(z) = H_0(z)(1-z) \quad \bar{H}_s(z) = H_s(z)(1-z)$$

for some polynomials $H_0(z), H_s(z)$. If we define the new variable $z := (I-\Pi)x$ we have that this satisfies the following equation

$$z^+ = \{I + K_0(\Pi) + (I-\Pi)\sum_{s=1}^{g} \bar{K}_s(\Pi)(I+E_s)H_s(\Pi)\}z$$

In order to analyze the asymptotic properties of the $z(t)$ it is convenient to introduce the matrix

$$P(t) := \mathbb{E}[z(t)z^T(t)]$$

After some computations we obtain that

$$\begin{aligned} P^+ &= \alpha(\Pi)P\alpha(\Pi^T) + \\ &\quad + \sum_{s=1}^{g} \bar{K}_s(\Pi)\Lambda_s \mathrm{diag}\{H_s(\Pi)PH_s(\Pi^T)\}\bar{K}_s(\Pi^T)\,, \end{aligned}$$

where $\Lambda_s := \mathbb{E}[E_s^2]$ is diagonal and where

$$\alpha(z) := 1 + K_0(z) + (1-z)\sum_{s=1}^{g} \bar{K}_s(z)H_s(z)$$

and

$$\mathrm{diag}\{Q\} := \mathrm{diag}\{q_{1,1},\dots,q_{N,N}\}.$$

We consider now the particular case in which $H_s(z) = z^{l_s}$. Let $w(t) = \mathrm{trace}\ P(t)$ and $K_s(z) := (1-z)\bar{K}_s(z)$ and assume that $\mathbb{E}[E_s^2] = \delta_s^2 I$. After some computations we obtain that $w(t)$ satisfies the following convolution equation

$$\begin{aligned} w(t) &= \mathrm{trace}\ \alpha(\Pi)^t P_0 \alpha(\Pi^T)^t + \\ &\quad + \sum_{i=0}^{t-1}\left(\sum_{s=1}^{g}\delta_s^2 ||\alpha(z)^i K_s(z)||^2\right) w_{t-1-i}\,, \end{aligned}$$

where $||\cdot||$ maps a polynomial $q(z) = \sum_{i=0}^{N-1} q_i z^i$ in the ring $\mathbb{R}[z,z^{-1}]/(z^N-1)$ to the number

$$||q(z)||^2 := \sum_{i=0}^{N-1} q_i^2$$

By defining the following power series

$$W(\xi) := \sum_{t=0}^{\infty} w(t)\xi^{-t},$$

$$A(\xi) := \sum_{t=0}^{\infty}\sum_{s=1}^{g}\delta_s^2||\alpha(z)^t K_s(z)||^2\xi^{-t}$$

$$B(\xi) := \sum_{t=0}^{\infty}(\mathrm{trace}\ \alpha(\Pi)^t P_0 \alpha(\Pi^T)^t)\xi^{-t}$$

we obtain that

$$W(\xi) = B(\xi) + \xi^{-1}W(\xi)A(\xi)$$

and so

$$W(\xi) = (1-\xi^{-1}A(\xi))^{-1}B(\xi)\,.$$

From the properties of circulant matrices (namely they can be digitalized by an orthogonal matrix and the eigenvalues are the Fourier transform of the coefficients) we can argue that

$$A(\xi) = \frac{1}{N}\sum_{h=0}^{N-1}\frac{\left(\sum_{s=1}^{g}\delta_s^2|K_s(e^{j\frac{2\pi}{N}h})|^2\right)\xi}{\xi - |\alpha(e^{j\frac{2\pi}{N}h})|^2}\,.$$

Example 4: Consider again Example 1 and assume that beyond the exact communication link $i \leftarrow i+1$ we also have noisy data transmission. For instance we consider the case in which $g = 1$ and $\bar{H}_1(z) = 1-z$. In this case we have that

$$x^+ = \{I + k(I-\Pi) + \bar{K}(\Pi)(I+E)(I-\Pi)\}x$$

and

$$\begin{aligned} z^+ &= \{I + k(I-\Pi) + (I-\Pi)\bar{K}(\Pi)(I+E)\}z \\ &= \alpha(\Pi)z + (I-\Pi)\bar{K}(\Pi)Ez\,, \end{aligned}$$

where $\alpha(z) = 1 + k(1-z) + (1-z)\bar{K}(z)$. If we impose that

$$\alpha(z) = \frac{1}{N}\sum_{i=0}^{N-1} z^i\,,$$

for example imposing,

$$k = \frac{1-N}{N} \quad \bar{K}(z) = \sum_{i=1}^{N-2}\frac{i+1-N}{N}z^i$$

then we have that

$$\alpha(e^{j\frac{2\pi}{N}h}) = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0\,. \end{cases}$$

This implies that

$$K(z) = \bar{K}(z)(1-z) = \alpha(z) - 1 - k(1-z)$$

that yields

$$K(e^{j\frac{2\pi}{N}h}) = \begin{cases} 0 & \text{if } h = 0 \\ -\frac{1}{N} - \frac{N-1}{N} e^{j\frac{2\pi}{N}h} & \text{if } h \neq 0 \,. \end{cases}$$

After some computations we obtain that

$$A(\xi) = \frac{\delta^2}{N} \sum_{h=1}^{N-1} |K(e^{j\frac{2\pi}{N}h})|^2 = \delta^2 f(N)$$

where $f(N) = (N^2 - 3N + 2)/N^2$. Notice moreover that, since $v^T P_0 v = 0$ and since $\alpha(\Pi)^2 = \alpha(\Pi)$, we obtain

$$B(\xi) = \text{trace } P_0 = w(0)$$

and so

$$W(\xi) = \frac{w(0)}{1 - \delta^2 f(N) \xi^{-1}} \,.$$

This implies $w(t) = (w(0))(\delta^2 f(N))^t$ converging to zero exponentially with rate $\delta^2 f(N)$.

This shows that for small δ the convergence rate is much better than what obtained without noisy data transmission (9). More precisely, suppose the our goal is to have convergence of the initial states $x_i(0) \in [-M, M]$ to a target configuration $x_i(\infty) \in [\alpha - \epsilon, \alpha + \epsilon]$ where α is a constant depending only on the initial condition $x(0)$ and ϵ describes the desired agreement precision. This is a "practical stability" requirement and it is the only goal achievable through finite data rate transmission. The parameter $C := M/\epsilon$ is called the contraction rate. We assume that the exact data transmissions are substituted by transmissions of precision ϵ uniformly quantized data. In this framework it is known [5] that each uniform quantizer needs C different levels and so the transmission of its data needs an alphabet of C different symbols. On the other hand (see [5]) each logarithmic quantizer needs

$$\frac{2 \log C}{\log \frac{1+\delta}{1-\delta}}$$

different symbols. Let $\delta = 1/2$. We know that the strategy proposed in this example allows a convergence rate equal to $\rho \simeq 1/2$. The total number of symbols L_{tot} which need to be transmitted for obtaining the agreement in this way is

$$L_{tot} = NC + \frac{2}{\log 3} N(N-2) \log C \,.$$

Without the logarithmic quantizers we need only $L_{tot} = NC$ symbols but we obtain a convergence rate

$$\rho \simeq 1 - \frac{2\pi^2}{N^2} \,.$$

Observe that for large C the total number of symbols L_{tot} in the two cases are slightly different, but the improvement in terms of rate of convergence is tremendous.

If we assume that $N = 2^\nu$ and we take as an alternative method the one proposed in example 3, it can be shown that in this way we obtain a convergence rate

$$\rho \simeq 1 - \frac{\pi^2}{2 \log N} \,,$$

with the total number of symbols $L_{tot} = CN \log N$. Also in this case it is clear that in some situations the technique based on the use of logarithmic quantizers proposed above presents better convergence performance with the use of a comparable total number of symbols.

V. CONCLUSIONS

We have showed that symmetries yield rather slow convergence to the consensus. In particular for such networks we have computed a tight bound for the convergence rate. We also studied control performance when agents exchange logarithmically quantized data. It has been shown that adding such links in networks with symmetries improves the convergence rate to the consensus with little growth of the required bandwidth.

The application of the techniques of control under communication constraints could be brought much further. For instance, instead of using logarithmic quantizers, the use of the quantizers with memory proposed in [8], [9] will yield much more efficient solutions. These issues will be the subject of our future investigations.

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