# Performance Analysis of Linear Estimators with Unknown Changes in Sensors Characteristics

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***Abstract*— Minimum variance state estimation for linear time-invariant systems with Gaussian state and measurement noise is achieved by the Kalman filter. This estimator is known to be robust to model uncertainties, however, it relies upon the knowledge of the measurement covariance. This is a serious limitation when measurements noise covariances change unpredictably because of external events, such as changes of lighting conditions, presence of smoke/fog, external magnetic fields, etc. In this paper, we consider and analyze a three stage estimation algorithm comprising of: 1) *Covariance estimation*, estimating the accuracy of each sensor; 2) *Measurement gating*, rejecting measurements until a new accuracy estimate is provided; and 3) the *Kalman filter*, estimating the state and its error covariance.**

**The main results of this paper are estimation error characterizations of the proposed three stage filter when the measurement noise covariances undergo sudden and unknown changes. We consider both the single and multi-sensor scenarios and provide a complete analysis for scalar systems along with key insights and preliminary results for the vector setting.**

***Index Terms*— Adaptive Kalman Filtering, Outlier rejection, Performance Analysis, Multi-sensor Fusion.**

## I. Introduction

The recent years have witnessed a steady increase in the use of sensors, actuators and computing devices within a large number of physical systems, such as cars, airplanes, power plants, buildings. In particular, new and more complex sensors, such as video cameras, LIDAR, infrared arrays, are being installed in such systems to increase their level of automation, security and safety. The properties of such sensors are typically difficult to characterize as they are intimately dependent on the pre-processing algorithms that extract higher level features from raw data. For example, a video-based tracking algorithm will provide the position of people with an accuracy that strongly depends on external factors such as lighting conditions, the type and direction of motion of the people, weather conditions (if outside), to name a few. These type of events can significantly decrease the performance of such complex sensors, considering that the intensity and occurrence of such events is not known in advance. Most of the estimation algorithms rely heavily upon the knowledge of the sensor model to provide acceptable estimates, but it is characterizing the measurement statistics in all possible scenarios is not achievable.

Various methodologies have been developed over the past decades to deal with these types of situations. However, to the best of the authors' knowledge, no formal analysis is available to predict their behavior in the context of accuracy and degree of robustness to changes.

In this paper, we consider a three stage estimation algorithm capable to deal with sudden changes in the sensors' accuracy and we provide a theoretical analysis of its performance. Such an algorithm adds two extra components around any existing (legacy) filter (e.g. Kalman filter, Extended Kalman Filter, Unscented Kalman Filter, etc.) enabling the use of complex sensors in highly unstructured and changing environments, even if such changes occur in an unknown fashion. The use of a legacy filter at the core of the estimation algorithm, although it might appear as a limiting factor, it is very desirable in practice as it enables "retrofitting" existing systems as well as faster and cheaper integration. In this paper, we focus on characterizing the *mean-squared estimation error*, after a sensor changes its accuracy (or becomes faulty). Furthermore, we characterize the *degree of robustness* of the proposed scheme, namely the minimum number of sensors that need to maintain a nominal operation whenever an unknown change occurs, so that the mean-squared error is within a certain desired threshold.

### A. Related Work

One of the key research areas that deal with varying sensor characteristics during estimation is *Adaptive Kalman Filtering*. Seminal works (cf. [1], [2]) propose online techniques to determine the measurement and process noise covariance matrices. In the case of linear systems with Gaussian noise, these techniques rely on the whiteness property of the innovations sequence for the filter to perform optimally. The work [3] presents a self-tuning Kalman filter via stochastic approximations. However, stochastic approximations tend to be slow to converge and require a good initial guess. More recently, heuristics based on sample averages have been proposed in the literature, cf. [4], [5]. These techniques have been demonstrated to work even for non-linear systems involving the Extended Kalman Filter (see [6], [7]). In [8] and in [9], the authors have presented a fuzzy neural network in conjunction with adaptive extended Kalman Filter to enhance the learning of the measurement noise covariance. In [10], the authors demonstrate this technique on a low-cost INS/GPS system and show improvement in the navigation estimation accuracy. In [11], the authors integrate a fuzzy

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version of the outlier detection technique used in this paper with the adaptive measurement error covariance estimation framework and demonstrate results on experimental data from autonomous underwater vehicle trials.

Another key research area related to this work is *outlier rejection*. One of the most popular techniques in the area of estimation to detect and reject erroneous measurements is *gating*, known alternatively as the *chi-squared test*, described in [12], [13], [14]. This technique is widely used in performing data association, as shown in [15], [16]. From the performance point-of-view, [17] derives a modified Riccati equation that approximately quantifies the dependence of the estimation error covariance on parameters such as gate thresholds and probability of false alarms. Given a set of measurements, a desired false-alarm probability, and a signal-to-noise ratio, the celebrated Neyman-Pearson detector has been known to minimize the false dismissal probability ([18]). In the case when the measurement signal is much weaker than the noise, locally optimal detectors have been proposed in [19]. More recently, simultaneous estimation of state and fault statistics for linear time-varying stochastic systems has been reported in [20].

### B. Contributions

We consider the problem of estimating the state of a discrete-time, linear time-invariant dynamical system when the accuracy of one or more sensors changes abruptly and in an unknown fashion at a certain time instant of operation. The performance metric of interest is the mean-squared estimation error. Particularly, for the case of a scalar state, and for a single sensor, we provide analytical bounds on the expected estimation error covariance, which show a great improvement over the baseline case, i.e., the standard Kalman Filter without any outlier detection scheme and with the wrong sensor covariance. This bound enables estimation of the *settling time*, i.e., the time taken to reduce average estimation error to within a desired threshold. In the case of multiple sensors, we are able to compute reasonably tight bounds on the minimum number of working sensors needed at each time step to guarantee a pre-specified performance under changes of accuracy of the remaining sensors. Analytical bounds are validated via numerical simulation. Finally, we provide key insights and some preliminary results on the extension of the multi-sensor problem to the vector case.

Although the methodology (Covariance Estimation, Gating and Kalman Filter) considered in this paper are well established in literature, as well as some of their combinations (see, e.g., [10], [11]), we believe that our contribution lies in providing the first attempt to analyze this framework for the linear time-invariant case. We acknowledge the results are somehow preliminary, as related to a scalar system when only one type of event occurs, i.e. sensors can only change their accuracy from a nominal value to a "worst" accuracy, however we believe they are instrumental to understand the more general case and they provide interesting fundamental limitations for the general adaptive filtering scheme we consider. As it will be clear in the following, the analysis of this type of systems is rather complex as the measurement gating process makes the overall filter a state dependent stochastic switching system.

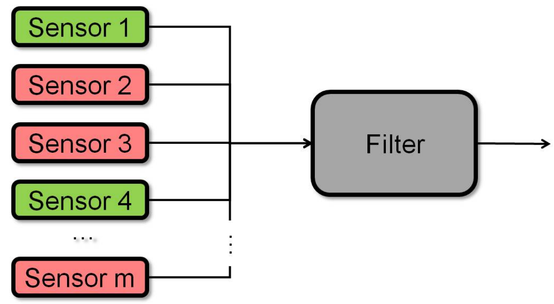


Fig. 1. Multiple failures/changes of accuracy among sensors.

### C. Organization of this paper

This paper is organized as follows: Section II describes the mathematical formulation of the problem and the technical approach adopted. Section III presents the analysis for the single sensor case for scalar systems. Section IV presents the analysis for the multiple sensor case for scalar systems, and also key insights into the vector case. Finally, Section V includes our conclusions and directions for future work. Due to lack of space, the proofs of all results have been included in the online technical report [21].

## II. Problem Set-up and Technical Approach

In this section, we present the mathematical formulation of the problem and the technical approach.

### A. Problem Formulation

Consider the discrete evolution of a linear time-invariant dynamical system given by the following equations:

$$x_{t+1} = Ax_t + \nu_t, \tag{1}$$
$$y_{j,t} = C_j x_t + v_{j,t}\,, \quad \forall j \in \{1,\ldots,m\}\,, \tag{2}$$

where $x_t \in \mathbb{R}^{n_x}$ is the state of the system, $y_t \in \mathbb{R}^{n_y}$ is the measurement vector, $\nu_t \sim \mathcal{N}(0,Q)$ and $v_t \sim \mathcal{N}(0,R_t)$ are zero mean, Gaussian random vectors at each time. In this set-up, the entries of the covariance matrix $R_t$ can vary with time, however, we will assume that its dynamics are not known by the filter. The measurement covariance matrix for the sensors has a nominal value $R_{j,t} := R_{\text{nom}}$, and at time $t = 0$, the covariance of some of the $m$ sensors increases to a value of at most $\eta$ times, where $\eta > 1$ is the accuracy change parameter. Although the analysis is presented for the case when all sensors have the same accuracy, it can be easily extended to the case of different accuracies for each sensor.

We consider a Kalman Filter as the core estimator, augmented with methods to detect measurements whose statistics have changed and to estimate the new measurement covariance. The goal of this paper is to analyze the *performance* of this estimation framework when applied to a system governed by (1) and (2). The measure of performance is the estimation error covariance $P_t := \mathbb{E}[(x_t - \hat{x}_t)(x_t - \hat{x}_t)'|\mathbf{Y}_t]$, where $\mathbf{Y}_t$ is the set of all measurements up to time $t$ and

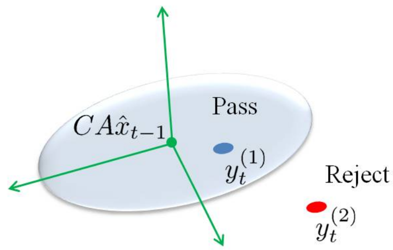


Fig. 2. Illustrating Measurement gating. The shaded region, centered at $CA\hat{x}_{t-1}$, corresponds to the gate with threshold equal to $\beta < 1$. The measurement $y_t^{(1)}$ *passes* the gating test if it lies inside the shaded region. Otherwise, the measurement *fails* the test, see measurement $y_t^{(2)}$.

$\hat{x}_t := \mathbb{E}[x_t|\mathbf{Y}_t]$ is the state estimate at time $t$. We address the following two cases:

1) *Single sensor case:* Given a change of accuracy of the sensor by a factor of $\eta$ (this is feasible in the scalar case), determine the *settling time*, i.e., time taken for the system to reduce the (average) estimation error covariance to within a given $\epsilon > 0$.
2) *Multi-sensors case:* Given a requirement of the (average) estimation error covariance to be bounded by $\epsilon > 0$, determine the minimum number of sensors $m_w$ that must retain their nominal accuracy.

As the dynamics of the three stage estimator considered here are very complex to be analyzed in the general case, in this paper we focus on the case when the state $x$ as well as the measurements $y_j$ are scalars, and provide some initial comments on the vector case, which will be the topic of further research.

### B. Technical Approach

The proposed methodology is summarized in Algorithm 1. To simplify the presentation, the algorithm is defined for the case when the change of sensor accuracy occurs at time $t = 0$. The first step consists of performing covariance estimation [7]. The method we adopt requires setting an estimation window $w > 1$. Up to time $t \leq w$, the estimate of $\hat{R}_t$ is the nominal value, and for $t > w$, one computes the moving average of the covariance of innovations $y_\tau - CA\hat{x}_{\tau-1}$ in the interval $\tau \in [t-w, t-1]$, and subtracts the term $C(AP_{t-1}A' + Q)C'$ from it. This equation is based upon the principle that the innovation sequence is white for an optimally functioning Kalman filter. Therefore, in the limit as $w \to +\infty$, and if the filter is operating optimally, the equation becomes exact.

The second step is to perform measurement gating [14], shown in Figure 2. At a time instant $t$, one has the predicted value of the measurement $CA\hat{x}_{t-1}$, the covariance of which equals $C(AP_{t-1}A' + Q)C' + \hat{R}_{j,t}$. Then, the true measurement $y_t$ will be in the following region

$$\mathcal{V}_t = \{y_t : (y_t - CA\hat{x}_{t-1})'(C(AP_{t-1}A' + Q)C' + \hat{R}_{j,t})^{-1} (y_t - CA\hat{x}_{t-1}) \leq \beta^2\} \quad (3)$$

with a probability determined by the gate threshold $\beta$ (sometimes referred to as the "number of sigmas"). The region

**Algorithm 1** Covariance estimation and Measurement gating

**Input:** $w, \beta, \{\hat{x}_k\}_{k=t-1}^{t-w}, \{y_{j,k}\}_{k=t}^{t-w}, P_{t-1}, \hat{R}_{j,0} = R_{j,nom}$.

1) Measurement Gating:

$$\gamma_{j,t} = \begin{cases} 1 & \text{if } r_j(t,0)(C_j(AP_{t-1}A' + Q)C_j' + \\ & \quad \hat{R}_{j,t-1})^{-1} r_j(t,0)' \leq \beta\,, \\ 0 & \text{otherwise.} \end{cases}$$

2) Multi-sensor Fusion Kalman Filter:

$$P_t^{-1} = (AP_{t-1}A' + Q)^{-1} + \sum_{j=1}^{m} \gamma_{j,t} C_j' \hat{R}_{j,t-1}^{-1} C_j, \quad (4)$$

$$\hat{x}_t = P_t((AP_{t-1}A' + Q)^{-1} A\hat{x}_{t-1} + \sum_{j=1}^{m} \gamma_{j,t} C_j' \hat{R}_{j,t-1}^{-1} y_{j,t}).$$

3) Estimate sensor accuracy:
Let $r_j(k) = y_{j,k} - C_j\hat{x}_k$

$$\hat{R}_{j,t} = \begin{cases} R_{j,nom} & \text{if } t \leq w \\ \frac{1}{w}\sum_{k=t-w+1}^{t} r_j(k) r_j(k)' + C_j P_t C_j' & \text{o.w.} \end{cases} \quad (5)$$

**Output:** $\hat{x}_t, P_t, \hat{R}_{j,t}$

defined by (3) is called gate or validation region, where the semi-axes of this ellipsoid are the square roots of the eigenvalues of $\beta^2(C(AP_{t-1}A' + Q)C' + \hat{R}_{j,t})$.

The third and final step comprises of a multi-sensor fusion for a Kalman filter. The covariance of the estimation error and the estimate are computed using the *inverse covariance* form [23], as shown in (4). This methodology is illustrated in Figure 3 for the case of a single sensor.

The distinguishing feature of our set-up, when compared to the works in the area of estimation over packet dropping links (e.g. see [22]) is that their formulation assumes that the random variables $\gamma_i$'s are generated *independently* out of a given distribution (typically Bernoulli) at each time. In our framework, the random variables $\gamma_i$'s are clearly dependent on $\hat{x}$ and $P$. Furthermore, we aim at characterizing performance metrics such as settling time, and minimum sensor suite design. Similar performance computation has been performed in [17] via a modified approximate Riccati equation. However, their work does not consider the above metrics which we characterize.

## III. Scalar Case: Single Sensor

In this section, we present analytical results for the case when there is only one scalar sensor measurement $y_t$. In the following, we will denote scalars with lower case letters.

Let us consider the following scenario: at time $t = 0$, the estimation error covariance is $p_0 > 0$, and the nominal measurement error covariance is $r_{\text{nom}}$. At $t = 1$, the measurement error covariance becomes equal to $\eta r_{\text{nom}}$, for some $\eta > 1$. Note that since the filter does not run optimally (due to the change in the value of $r$), the estimation error covariance is no longer given by $p$. The filter would run optimally only

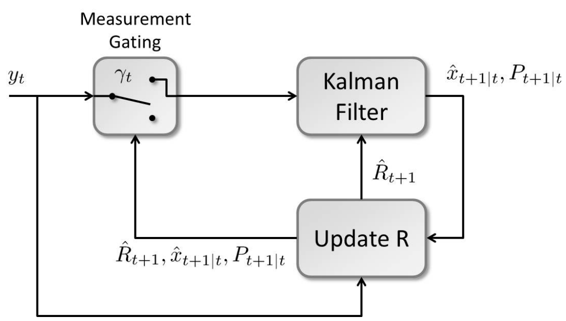


Fig. 3. Illustration of Algorithm 1 on the single sensor case.

when the $R$-estimation step in Algorithm 1 converges to the true value. Therefore, we introduce the following assumption to make the analysis tractable.

**Assumption III.1** (Finite time convergence of R-estimation)**.** *The R-estimation procedure in* (5) *converges in at most $w$ time steps, i.e., after time $t = w$, $\hat{r}_t = \eta r_{\text{nom}}$.*

While this assumption may appear restrictive, it has been observed to hold reasonably consistently in simulation. This assumption makes Algorithm 1 equivalent to decoupling the $R$-estimation step from the rest of the Algorithm. Although the algorithm has been presented such that the accuracy change occurs at time $t = 0$, one can easily generalize the statement and the analysis to include the case when the accuracy change occurs at a different time $t_s > 0$. The only change is that in the time interval $[t_s, t_s+w]$, the filter works with the nominal covariance, i.e., $\hat{r}_t = r_{\text{nom}}$. This general case has been implemented in the simulation result shown in Figure 4.

To solve the single sensor problem, it suffices to determine an upper bound on the estimation error covariance, i.e., $\mathbb{E}[e_t^2]$, where $e_t := \hat{x}_t - x_t$. Such a bound on the estimation error covariance is provided in the following result.

**Theorem III.1** (Single sensor case)**.** *Suppose that Algorithm 1 is applied to an open-loop stable (i.e., $a < 1$ ) system* (1) *and* (2) *with $n_x = n_y = 1$. Under Assumption III.1,*

1) *If $t \leq w$, where $w$ is the estimation window, then*

$$\mathbb{E}[e_t^2] \leq \frac{q + \lambda_{\max}\beta^2 q/(1-a^2)}{1-a^2},$$

*where*

$$\lambda_{\max} := \operatorname{erf}\left(\frac{\beta}{\sqrt{2}}\sqrt{\frac{q/(1-a^2)+\sigma}{q+\eta\sigma}}\right),$$

*and the expectation is with respect to the random variables $\gamma_t, w_t, v_t$, $\forall t \in \{1, 2, \ldots, w\}$.*

2) *For $t > w$,*

$$\mathbb{E}[e_t^2] \leq -\zeta + \left(\frac{\left(\frac{\bar{d}-\zeta\bar{c}}{\zeta\bar{c}+\bar{a}}\right)^{t-w}}{\zeta+\bar{p}_w} + \frac{\bar{c}}{\zeta\bar{c}+\bar{a}}\left(\frac{1-\frac{(\bar{d}-\zeta\bar{c})^{t-w}}{(\zeta\bar{c}+\bar{a})^{t-w}}}{1-\frac{(\bar{d}-\zeta\bar{c})}{(\zeta\bar{c}+\bar{a})}}\right)\right)^{-1},$$

*where*

$$\begin{aligned}
&\bar{a} := a^2, \quad \bar{b} := q, \quad \bar{c} := a^2/(\eta\sigma),\\
&\bar{d} := q/(\eta\sigma) + 1,\\
&\zeta := (\bar{d} - \bar{a} + \sqrt{(\bar{d}-\bar{a})^2 + 4\bar{b}\bar{c}})/(2\bar{c}),\\
&\bar{p}_w := (q + \lambda_{\max}\beta^2 q/(1-a^2))/(1-a^2).
\end{aligned}$$

3) *In the limit as $t \to +\infty$,*

$$\lim_{t\to\infty} \mathbb{E}[e_t^2] \leq \frac{\sqrt{(\bar{d}-\bar{a})^2 + 4\bar{b}\bar{c}} - (\bar{d}-\bar{a})}{2\bar{c}}.$$

This result immediately provides us with the following bound on the settling time $T_s$, which is the time interval such that $\mathbb{E}[e_{w+T_s}^2] \leq \epsilon$, for a specified feasible $\epsilon$.

**Corollary III.1** (Settling time)**.** *Under Assumption III.1, given an $\epsilon > (\sqrt{(\bar{d}-\bar{a})^2 + 4\bar{b}\bar{c}} - (\bar{d}-\bar{a}))/(2\bar{c})$, the steady state bound in case 3) of Theorem III.1, the settling time is at most*

$$T_s \leq \ln\left(\frac{\zeta+\epsilon}{\zeta+\bar{p}_w}\left(1 + \frac{\bar{c}}{\sqrt{(\bar{d}-\bar{a})^2+4\bar{b}\bar{c}}}\right)\right) \Big/ \ln\left(\frac{\zeta\bar{c}+\bar{a}}{\bar{d}-\zeta\bar{c}}\right).$$

This result follows by simply setting the right hand side of statement 2) of Theorem III.1 to be less than $\epsilon$.

We simulated the scalar system with the parameters $a = 0.9, c = 1, w = 30, q = r_{\text{nom}} = 1, \eta = 1000$, and with the initial estimation error covariance $p_0$ equal to the solution of the discrete time Algebraic Riccati Equation for the nominal covariance of the sensor. At time $t = 60$, the sensor changes accuracy. We compare the theoretical bounds from Theorem III.1 with numerically determined error covariance averaged over 100 Monte Carlo runs. The result is summarized in Figure 4.

The numerically determined curve does not always lie below the theoretical bound, and the reason is that Theorem III.1 holds under Assumption III.1. While we observe that the $R$-estimation step almost converges, the value of $R$ is not exactly a constant and shows minor fluctuations with time. Nevertheless, the theoretical bound serves as a very good approximation to the numerical curve.

## IV. Scalar Case: Multiple Sensors

In this section, we consider the case of multiple sensors and a scalar system. For ease of presentation, we will consider the case of identical sensors which, with an extra level of book-keeping, can be easily extended to non-identical sensor characteristics. We address the case that for time $t > 0$, at least $m_w \geq 1$ sensors retain their nominal characteristics, while the error covariance of at most $m - m_w$ sensors may increase by a factor of $\eta > 1$. Even for an open-loop unstable system, due to sensor redundancy, the filter will continue to remain stable. However, the estimation accuracy will now be governed by $m_w$.

Without any loss of generality, let us assume that the first $m_w$ sensors are in working condition. After a time given by the R-estimation window $w$, the values of $\hat{r}_{j,t}$ for each of the $m - m_w$ sensors are expected to be reasonably close to $\eta r_{\text{nom}}$,

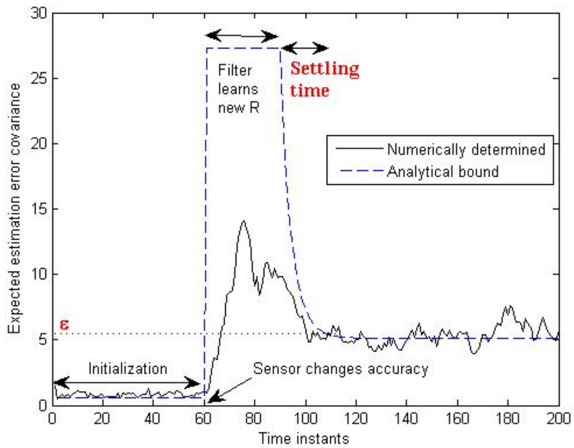


Fig. 4. Time evolution of the expected estimation error covariance. The change of accuracy occurs at time $t = 60$. The theoretical upper bound is given by Theorem III.1 and the numerically determined evolution is given by the black solid line.

for a sufficiently large $w$. Therefore, for ease of theoretical computation, we introduce the following assumption.

**Assumption IV.1** (Multi-sensor R-estimation)**.** *For each of the $m - m_w$ sensors, the R-estimation procedure in (5) converges in at most $w$ time steps, i.e., after time $t = w$, $\hat{r}_{j,t} = \eta r_{\text{nom}}, \forall j \in \{m_w + 1, \ldots, m\}$.*

In our analysis of Algorithm 1, we characterize upper and lower bounds on the estimation error covariance and we use them to derive the minimum number of sensors needed to guarantee a desired level of performance, i.e., given an $\epsilon > 0$, the steady state estimation error covariance is at most $\epsilon$. For a lower bound on the estimation error, in light of Assumption IV.1, we will consider the *best-case scenario* in which after the time window of $w$, the filter learns the changed accuracy $\eta\sigma$ of the $m - m_w$ sensors. For the corresponding upper bound, we will consider the *worst-case scenario* in which the sensors that changed accuracy are simply *gated out*, i.e., ignored.

Then, the following result can be established.

**Theorem IV.1** (Multiple sensors case)**.** *Under Assumption IV.1,*

$$\frac{\sqrt{(kq + \sigma - \sigma a^2)^2 + 4\sigma q k a^2} - (kq + \sigma - \sigma a^2)}{2ka^2} \leq \lim_{t \to +\infty} p_t \leq \frac{\sqrt{(m_w q + \sigma - \sigma a^2)^2 + 4\sigma q m_w a^2} - (m_w q + \sigma - \sigma a^2)}{2m_w a^2},$$

*where $m_w \geq 1$ is the number of sensors retaining their accuracy, $m$ is the total number of sensors and $k := m_w + (m - m_w)/\eta$.*

We simulated the system with the parameters $a = 0.9, c_j = 1, w = 30, q = 1, r_{\text{nom}} = 0.1, \eta = 5, m = 10$. We numerically determine the estimation error covariance resulting out of Algorithm 1 in its entirety in Figure 5. The numerical value of the estimation error has been averaged over 20 Monte Carlo trials. A comparison with the analytical bounds from Theorem IV.1 shows that while the lower bound condition is obeyed at all instances, due to the fluctuations in the covariance estimation of the working sensors, the upper bound is not always valid.

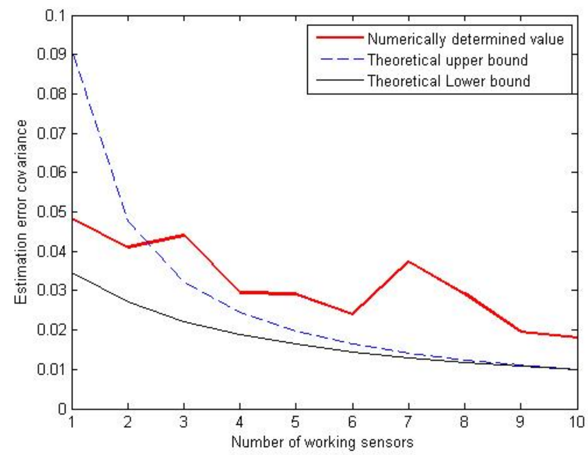


Fig. 5. Evolution of the estimation error covariance with the number of working sensors $m_w$ when applying Algorithm 1 in its entirety in the multiple sensors case.

To have a fair comparison with Theorem IV.1, we performed another simulation in which the covariances of working sensors were passed exactly to the filter while those of the faulty ones were learned using the covariance estimation step. The results are reported in Figure 6. From this figure, we can see that the empirical value of the estimation error covariance lies between the upper and lower bounds, thus verifying our analysis. In fact, the gap with the theoretical lower bound reduces as the number of working sensors increase, as is expected. Note also that the bounds are not very conservative and therefore, are useful to predict the performance of the proposed scheme to multiple changes of accuracy of the sensors.

**Remark IV.1** (Non-identical sensors)**.** *The analysis in this section can be extended to the case of non-identical sensors. In that setting, the performance, in general, will depend on the type (or the accuracy) of the nominally working sensors rather than their number $m_w$. Further, the design problem now becomes conceptually similar to a knapsack problem in combinatorial optimization [24], in which the goal will be to select the* best *set of sensors that satisfy a given performance requirement.*

**Remark IV.2** (Extension to the vector case)**.** *The analysis in this section can be extended to the vector case by considering the maximum eigenvalue $\lambda_{\max}(P_t)$ as the performance*

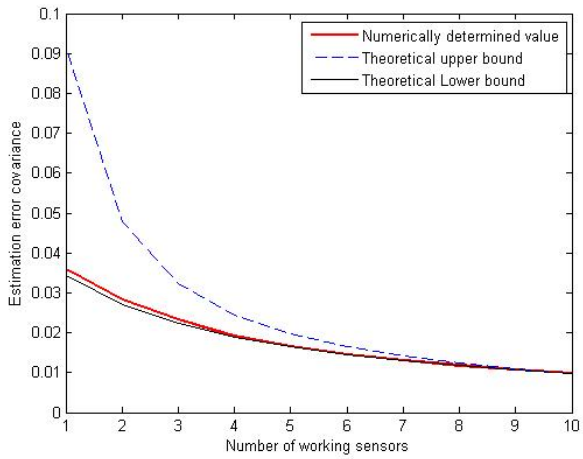


Fig. 6. Evolution of the estimation error covariance when the sensor covariances of the working sensors are known exactly by the filter in the multiple sensors case.

*metric. We can show (cf. [21]) that*

$$\lambda_{\max}(P_t) = \frac{\lambda_{\max}^2(A)\lambda_{\max}(P_{t-1}) + \lambda_{\max}(Q)}{\lambda_{\min}(\sum_{j=1}^{m_w} C_j{}' R_j^{-1} C_j)(\lambda_{\max}^2(A)\lambda_{\max}(P_{t-1}) + \lambda_{\max}(Q)) + 1}.$$

*Substituting $z_t := \lambda_{\max}(P_t)$, we obtain a linear rational recurrence in $z_t$, which has identical structure as in the scalar case, and therefore, a similar bound may be computed. This bound becomes useful when $\lambda_{\min}(\sum_{j=1}^{m_w} C_j{}' R_j^{-1} C_j) > 0$, which is equivalent to having every mode of the system being* observed *by some sensor at each time.*

## V. Conclusion and Future Directions

In this paper, we analyzed a framework comprising of a covariance estimator and an outlier rejection module, using the measurement gating, "wrapped around" a Kalman filter. This was analyzed for a linear time invariant system when sensors' accuracy varies in an unknown fashion. We presented analytical bounds on the evolution of the estimation error covariance for the scalar system for both the single sensor and the multiple sensors cases. In the multiple sensors case, our bounds provide a design criterion in the sense of the minimum number of reliable sensors needed in order to satisfy a performance requirement, i.e., the estimation error to be less than a specified threshold. We also provided key insights in analyzing the vector case. Simulation results verifying all of the analytical bounds were also presented.

An immediate future direction is to derive a complete characterization of the vector case in the non-linear setting with the Extended Kalman Filter as the core estimator. Also, of interest, is the extension to situations where the accuracy of sensors can change at multiple time instances.

## References


[1] R. K. Mehra, "On the Identification of Variances and Adaptive Kalman Filtering", in *IEEE Transactions of Automatic Control*, (15)2, pp.175–184, 1970.

[2] F. Gustafsson, "Adaptive Filtering and Change Detection", John Wiley and Sons, 2000.

[3] N. Madjarov and L. Mihaylova, "Kalman Filter Sensitivity with respect to Parametric Noises Uncertainty", Kybernetika, (32)3, pp. 307–322, 1996.

[4] F. Busse, and J. How, "Demonstration of Adaptive Extended Kalman Filter for Low Earth Orbit Formation Estimation using CDGPS", in *Institute of Navigation GPS Meeting*, 2002.

[5] J. Wang, M. Stewart, and M. Tsakiri, "Online Stochastic Modeling for INS/GPS Integration", in *ION GPS-99 proceedings*, pp. 1887–1895, 1999.

[6] A. H. Mohamed and K. P. Schwarz, "Adaptive Kalman Filtering for INS/GPS", in *Journal of Geodesy*, vol. 73, pp. 193–203, 1999.

[7] W. Ding, J. Wang, C. Rizos and D. Kinlyside, "Improving Adaptive Kalman Estimation in GPS/INS Integration", in *The Journal of Navigation*, vol. 60, pp. 517–529, 2007.

[8] P. J. Escamilla and N. Mort, "A hybrid Kalman filter-fuzzy logic multisensor data fusion architecture with fault tolerant characteristic", in *Proceedings of the 2001 International Conference on Artificial Intelligence*, pp. 361-367, 2001.

[9] D-J. Jwo and H-C. Huang, "GPS Navigation Using Fuzzy Neural Network Aided Adaptive Extended Kalman Filter", in *Proceedings of the IEEE Conference on Decision and Control and European Control Conference*, 2005.

[10] C. Hide, T. Moore and M. Smith, "Adaptive Kalman Filtering for Low-cost INS/GPS", in *The Journal of Navigation*, (56)1, 2003.

[11] D. Loebis, R. Sutton, J. Chudley and W. Naeem, "Adaptive tuning of a Kalman filter via fuzzy logic for an intelligent AUV navigation system", in *Control Engineering Practice*, Vol. 12, pp. 1531–1539, 2004.

[12] B. D. Brumback and M. D. Srinath, "A Chi-square test for Fault Detection in Kalman Filters", in *IEEE Transactions on Automatic Control*, (32)6, 1987.

[13] Y. Bar-Shalom, X. R. Li, T. Kirubarajan, "Estimation with Applications to Tracking and Navigation", Wiley-Interscience, 2001.

[14] Y. Bar-Shalom, P. Willett and X. Tian, "Tracking and Data Fusion", YBS Publishing, Storrs, CT, 2011.

[15] S-W. Yeom, T. Kirubarajan and Y. Bar-Shalom, "Track Segment Association, Fine-Step IMM and Initialization with Doppler for Improved Track Performance", in *IEEE Transactions on Aerospace and Electronic Systems*, (40)1, pp. 293–309, 2004.

[16] T. Fortmann, Y. Bar-Shalom and M. Scheffe, "Sonar Tracking of multiple targets using joint probabilistic data association", in *IEEE Journal of Oceanic Engineering*, (8)3, pp. 173–184, 1983.

[17] T. Fortmann, Y. Bar-Shalom, M. Scheffe and S. Gelfand, "Detection thresholds for tracking in clutter – A connection between estimation and signal processing", in *IEEE Transactions on Automatic Control*, (30)3, pp. 221–229, 1985.

[18] W. W. Peterson, T. G. Birdsall, and W. C. Fox, "The theory of signal detectability", in *IRE Transactions on Information Theory*, 4, pp. 171 – 212, 1954.

[19] J. Capon, "On the asymptotic efficiency of locally optimal detectors", in *IRE Transactions on Information Theory*, (7)2, pp. 67 – 71, 1961.

[20] K. Khemin, F.B. Hmida, J. Ragot and M. Gossa, "Novel optimal recursive filter for state and fault estimation of linear stochastic systems with unknown disturbances", Int. J. Applied Mathematics and Computer Science, AMCS, 2011, Vol.21, No.4, pp.629-637.

[21] S. D. Bopardikar, A. Speranzon, S. Zhang, and B. Sinopoli, "Performance Analysis of Linear Estimators with Unknown Changes in Sensors Characteristics", Technical report. Available online at `http://sites.google.com/site/albspe/publications`

[22] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman Filtering with Intermittent Observations", in *IEEE Transactions on Automatic Control*, (49)9, 2004.

[23] H. Durrant-Whyte, "Multi Sensor Data Fusion", Lecture Notes version 1.2. Available at `http://www.acfr.usyd.edu.au/pdfs/training/multiSensorDataFusion/dataFusionNotes.pdf`, 2001.

[24] H. Kellerer, U. Pferschy and D. Pisinger, "Knapsack Problems", Springer, 2004.