On Decentralized Negotiation of Optimal Consensus ⋆

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Abstract

A consensus problem consists of finding a distributed control strategy that brings the state or output of a group of agents to a common value, a consensus point. In this paper, we propose a negotiation algorithm that computes an optimal consensus point for agents modeled as linear control systems subject to convex input constraints and linear state constraints. By primal decomposition and incremental subgradient methods, it is shown that the algorithm can be implemented such that each agent exchanges only a small amount of information per iteration with its neighbors.

Key words: Consensus, Optimal trajectory planning, Decentralized optimization, Model predictive control, Convex optimization

1 Introduction

The problem of cooperatively controlling systems composed of a large number of autonomous agents has attracted substantial attention in the control and robotics communities. An interesting instantiation is the consensus problem, see for example the recent survey paper Olfati-Saber et al. (2007) and the references therein. It consists of designing distributed control strategies such that the state or output of a group of agents asymptotically converges to a common value, a *consensus point.* The agents are typically modeled by identical first-order systems with no input constraints.

The main contribution of this paper is a decentralized negotiation algorithm that computes the optimal consensus point for a set of agents modeled as linear control systems. In this paper, the consensus point is a vector that specifies, for example, the position and velocity the agents shall converge to. Our approach allows us to incorporate constraints on the state and the input, which is not easily done for the traditional consensus algorithm, see the discussion in Marden et al. (2007). By primal decomposition and incremental subgradient methods we design a decentralized negotiation algorithm, in which each agent performs individual planning of its trajectory and exchanges only a small amount of information per iteration with its neighbors. We show that the cost of reaching the consensus point can be significantly reduced, by letting the agents negotiate to find an optimal or near optimal consensus point, before applying a control signal.

There has been a lot of research activity in this area, and a good starting point for related work is the recent survey paper Olfati-Saber et al. (2007). In particular, if the consensus point is a position and *fixed a priori* [1] (contrary to our approach, where the optimal consensus point is a decision variable) we get a so called rendezvous problem. For this type of problem, much work have been focused on establishing convergence to the fixed consensus point under different communication and visibility conditions, see for example Cortéz et al. (2006)

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[1] In the consensus literature, the consensus point is typically fixed in the sense that it is computed from the initial conditions using a simple rule, for example, the consensus point could be the average of the starting positions of the agents.

and the references therein. Furthermore, optimal control formulations have been used in papers that focus on the convergence of distributed model predictive control (MPC) based strategies to an *a priori* fixed equilibrium point. Dunbar and Murray (2006) propose a decentralized scheme where a given desired equilibrium point is asymptotically reached. The scheme requires coupled subsystems to update and exchange the most recent optimal control trajectories prior to each update step. Stability is guaranteed if each subsystem does not deviate too far from the previous open-loop trajectory. In Keviczky et al. (2006), the authors propose a strategy where each subsystem solves a finite time optimal control problem. The solution of the problem requires each subsystem to know the neighbors' model, constraints, and state. The strategy also requires the prior knowledge of an overall system equilibrium. Finally, a related distributed optimization problem, focused on formation flight, is considered in Raffard et al. (2004), where the decentralized algorithm is based on dual relaxation. Their approach differs from ours in that they do not consider the consensus problem and that they use dual relaxation instead of primal decomposition.

The outline of the paper is as follows. In Section 2, we formulate the optimal consensus problem. The novel distributed negotiation algorithm is presented in Section 3. Section 4 discusses some control strategies and shows a numerical example. Finally, the paper is concluded in Section 5.

2 Problem Formulation

Consider $N > 1$ agents whose dynamics are described by

$$\begin{aligned} x_i(t+1) &= A_i x_i(t) + B_i u_i(t) \\ z_i(t) &= C_i x_i(t)\,, \qquad i = 1,\dots,N\,, \end{aligned} \tag{1}$$

where $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times p_i}$, and $C_i \in \mathbb{R}^{s \times n_i}$ are observable and controllable. The vector $x_i(0) = x_i^0 \in \mathbb{R}^{n_i}$ is the initial condition and $z_i(t)$ is the performance output. We assume that the inputs are constrained according to

$$\left(u_i^\intercal(0), u_i^\intercal(1), \dots, u_i^\intercal(T)\right)^\intercal \in \mathcal{U}_i,\ i = 1,\dots,N, \tag{2}$$

where T is a (fixed) time horizon and $\mathcal{U}_i$ is a convex set. By using standard techniques from MPC, the constraint can encode magnitude and rate constraints on $u_i(t)$, as well as restrictions on linear combinations of the agent states (Maciejowski, 2002, Sec 3.2).

Definition 1 *Let θ lie in a compact and convex set $\Theta \subset \mathbb{R}^s$. The agents described by (1) reach* consensus [2] *at time T if*

$$z_i(T+k) = \theta,\ \text{for all}\ k \geq 0\ \text{and}\ i = 1,\dots,N,$$

with $u_i(T+k) = u_i(T)$, for all $k \geq 0$ and $i = 1,\dots,N$.

The objective is to find a consensus point $\theta \in \Theta$ and a sequence of inputs $\left(u_i^\intercal(0), u_i^\intercal(1), \dots, u_i^\intercal(T)\right)^\intercal \in \mathcal{U}_i$, with $i = 1,\dots,N$, such that consensus is reached at time T. The following cost function is associated to the i-th system:

$$\begin{aligned} V_i(z_i(t), u_i(t-1), \theta) \triangleq{}& (z_i(t) - \theta)^\intercal Q_i \,(z_i(t) - \theta) \\ &+ u_i(t-1)^\intercal R_i u_i(t-1), \end{aligned} \tag{3}$$

where $Q_i \in \mathbb{R}^{s \times s}$ and $R_i \in \mathbb{R}^{p_i \times p_i}$ are positive definite symmetric matrices that encode the cost of deviating from the consensus point and the cost of control energy for agent i. Let us introduce the following vectors:

$$\begin{aligned} \mathbf{x}_i &\triangleq \left(x_i^\intercal(1), x_i^\intercal(2), \dots, x_i^\intercal(T+1)\right)^\intercal \\ \mathbf{u}_i &\triangleq \left(u_i^\intercal(0), u_i^\intercal(1), \dots, u_i^\intercal(T)\right)^\intercal. \end{aligned}$$

Since

$$\mathbf{x}_i = \underbrace{\begin{pmatrix} A_i \\ A_i^2 \\ \vdots \\ A_i^{T+1} \end{pmatrix}}_{\mathbf{E}_i} x_i^0 + \underbrace{\begin{pmatrix} B_i & 0 & \dots & 0 \\ A_i B_i & B_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_i^T B_i & A_i^{T-1} B_i & \dots & B_i \end{pmatrix}}_{\mathbf{F}_i} \mathbf{u}_i\,,$$

we have $z_i(T) = C_i x_i(T) = \mathbf{H}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) = \theta$, where $\mathbf{H}_i \triangleq \begin{pmatrix} 0 & \dots & C_i & 0 \end{pmatrix}$. We also introduce $\mathbf{U}_i \triangleq A_i^{T+1} - A_i^T$ and $\mathbf{W}_i \triangleq \begin{pmatrix} A_i^T B_i & A_i^{T-1} B_i & \dots & B_i \end{pmatrix} - \begin{pmatrix} A_i^{T-1} B_i & A_i^{T-2} B_i & \dots & 0 \end{pmatrix}$. We now formulate the optimization problem,

$$\begin{aligned} \underset{\mathbf{u_1},\dots,\mathbf{u_N},\,\theta}{\text{minimize}} \quad & \sum_{i=1}^{N} \mathbf{V}_i(\mathbf{u}_i, \theta) && \text{(4a)} \\ \text{s.t.} \quad & \mathbf{H}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) = \theta,\ i = 1,\dots,N && \text{(4b)} \\ & \mathbf{U}_i x_i^0 + \mathbf{W}_i \mathbf{u}_i = 0,\ i = 1,\dots,N && \text{(4c)} \\ & \mathbf{u}_i \in \mathcal{U}_i,\ i = 1,\dots,N && \text{(4d)} \\ & \theta \in \Theta, && \text{(4e)} \end{aligned}$$

[2] By introducing a fixed offset, $\bar{\theta}_i$, one for each agent, it is possible to define a *consensus formation* relative to a global consensus point θ. The condition of consensus formation is that $z_i(T+k) = \theta + \bar{\theta}_i$, for all $k \geq 0$ and $i = 1,\dots,N$.

with the cost function

$$
\begin{aligned}
\mathbf{V}_i(\mathbf{u}_i, \theta) &\triangleq \sum_{t=1}^{T+1} V_i(z_i(t), u_i(t-1), \theta) \\
&= (\mathbf{C}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) - \mathbf{1}_{T+1} \otimes \theta)^\intercal \mathbf{Q}_i (\mathbf{C}_i(\mathbf{E}_i x_i^0 \\
&\quad + \mathbf{F}_i \mathbf{u}_i) - \mathbf{1}_{T+1} \otimes \theta) + \mathbf{u}_i^\intercal R_i \mathbf{u}_i,
\end{aligned}
$$

where [3] $\mathbf{Q}_i = \mathbf{I}_{T+1} \otimes Q_i$, $\mathbf{R}_i = \mathbf{I}_{T+1} \otimes R_i$, and $\mathbf{C}_i = \mathbf{I}_{T+1} \otimes C_i$. Notice that the constraint (4b) guarantees consensus at time T and (4c) guarantees that the consensus point is an equilibrium, i.e., $x_i(T) = A_i x_i(T) + B_i u_i(T)$. The constraint (4b) can potentially lead to infeasibility problems, but such problems can be mitigated by replacing the constraint with a penalty term in the objective, penalizing deviations from the consensus point at time T. Note, however, that due to assumption A2 infeasibility problems do not arise in our setup.

We make the following standing assumptions, which make the optimization problem (4) convex and feasible, and guarantee that the consensus point is an equilibrium.

A1: The matrices $Q_i \in \mathbb{R}^{s \times s}$ and $R_i \in \mathbb{R}^{p_i \times p_i}$, $i = 1, \ldots, N$, are positive definite and symmetric. The set Θ is convex and compact. The sets $\mathcal{U}_i$, $i = 1, \ldots, N$, are convex.

A2: For all $\theta \in \Theta$, $x_i \in \{y \in \mathbb{R}^{n_i} | C_i y = \theta\}$, and $i = 1, \ldots, N$, there exists $\mathbf{u}_i$ in the relative interior of $\mathcal{U}_i$ such that $z_i(T) = \theta$ and $x_i = A_i x_i + B_i u_i(T)$.

The optimization problem (4) is interesting for a multi-agent setting if the computations can be distributed among the agents and the amount of information that they need to exchange is limited. In the following we develop a negotiation algorithm to find the optimal consensus point, in which agents exchange only their current estimates of θ.

3 Distributed Negotiation

To distribute the computation of the optimal consensus point, we use primal decomposition in combination with an incremental subgradient method (Bertsekas et al., 2003). Let us start with defining $q_i(\theta)$ as follows

$$
\begin{aligned}
q_i(\theta) = \min_{\mathbf{u}_i} \quad & \mathbf{V}_i(\mathbf{u}_i) \\
\text{s.t.} \quad & \mathbf{H}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) = \theta \\
& \mathbf{U}_i x_i^0 + \mathbf{W}_i \mathbf{u}_i = 0 \\
& \mathbf{u}_i \in \mathcal{U}_i,
\end{aligned} \tag{5}
$$

where we have eliminated the dependence on θ in $\mathbf{V}_i(\mathbf{u}_i, \theta)$ using the constraint $\mathbf{H}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) = \theta$. The optimization problem (4) can be written as

$$
\begin{aligned}
\underset{\theta}{\text{minimize}} \quad & \sum_{i=1}^{N} q_i(\theta) \\
\text{s.t.} \quad & \theta \in \Theta\,.
\end{aligned} \tag{6}
$$

We then have the following result.

Proposition 2 *The cost function $q_i(\cdot)$ defined in (5) is a convex function. A subgradient λ_i for $q_i(\cdot)$ at θ is given by the Lagrange multipliers corresponding to the constraint $\mathbf{H}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) = \theta$.*

PROOF. By Lagrangian relaxation we can define

$$
L_i(\mathbf{u}_i, \theta, \lambda_i) = \mathbf{V}_i(\mathbf{u}_i) - \lambda_i^\intercal (\mathbf{H}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) - \theta)\,,
$$

where λ_i are Lagrange multipliers. We also introduce the dual function

$$
d_i(\lambda_i, \theta) = \min_{\mathbf{u}_i \in \tilde{\mathcal{U}}_i} \left\{ \mathbf{V}_i(\mathbf{u}_i) - \lambda_i^\intercal (\mathbf{H}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) - \theta) \right\},
$$

where $\tilde{\mathcal{U}}_i = \{u_i \in \mathcal{U}_i | \mathbf{U}_i x_i^0 + \mathbf{W}_i \mathbf{u}_i = 0\}$. Strong duality follows from Theorem 6.4.4 (p. 373) in Bertsekas et al. (2003), since

(1) the constraint $\mathbf{H}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) = \theta$ is linear in $\mathbf{u}_i$,
(2) assumption **A2** guarantees that there exists a solution in the relative interior of $\mathcal{U}_i$ to this equation,
(3) the function $\mathbf{V}_i(\cdot)$ and the set $\mathcal{U}_i$ are convex.

Hence, $q_i(\theta) = \max_{\lambda_i} d_i(\lambda_i, \theta)$. Consider two feasible points, $\theta^\dagger$ and $\theta^\ddagger$, and let $\lambda_i^\dagger$ be the Lagrange multipliers corresponding to the relaxed constraint for $\theta^\dagger$, then

$$
\begin{aligned}
q_i(\theta^\ddagger) &= \max_{\lambda_i} d_i(\lambda_i, \theta^\ddagger) \geq d_i(\lambda_i^\dagger, \theta^\dagger) + (\lambda_i^\dagger)^\intercal (\theta^\ddagger - \theta^\dagger) \\
&= q_i(\theta^\dagger) + (\lambda_i^\dagger)^\intercal (\theta^\ddagger - \theta^\dagger).
\end{aligned}
$$

Hence, by the definition of a subgradient, $\lambda_i^\dagger$ is a subgradient of $q_i(\cdot)$ at $\theta^\dagger$. Now $q_i(\theta^\ddagger)$ can be expressed as

$$
\begin{aligned}
q_i(\theta^\ddagger) &= \max_{\lambda_i} \left\{ d_i(\lambda_i, \theta) + \lambda_i^\intercal (\theta^\ddagger - \theta) \right\} \\
&= \max_{\lambda_i} \left\{ g(\lambda_i) + \lambda_i^\intercal \theta^\ddagger \right\},
\end{aligned}
$$

where $g(\lambda_i) = d_i(\lambda_i, \theta) - \lambda_i^\intercal \theta$ and $g(\lambda_i) + \lambda_i^\intercal \theta^\ddagger$ is affine in $\theta^\ddagger$. Since $q_i(\theta^\ddagger)$ is the pointwise maximum of a family of convex functions, $q_i(\theta^\ddagger)$ is convex. $\square$

[3] With $\mathbf{1}_{T+1}$ we denote the column vector with $T+1$ ones, with $\mathbf{I}_{T+1}$ the $T+1 \times T+1$ identity matrix, and with $\otimes$ the Kronecker matrix product.

The optimal consensus point can be computed in a distributed way using the incremental subgradient methods from optimization theory (Bertsekas et al., 2003). In this scheme, an estimate of the optimal consensus point is passed around between agents. Upon receiving an estimate from its neighbor, an agent solves the optimization problem (5) to evaluate its cost of reaching the suggested consensus point and to compute an associated subgradient (using Proposition 2). The agent then updates the consensus estimate via

$$\theta_{k+1} := \mathcal{P}_\Theta\{\theta_k - \alpha_h \lambda_{i,k}\} \tag{7}$$

and passes the estimate to the next agent. The algorithm proceeds iteratively. Here $\mathcal{P}_\Theta\{\cdot\}$ denotes the Euclidean projection on the set Θ, α_h is the stepsize, and $\lambda_{i,k}$ is the subgradient of $q_i(\cdot)$ at θ_k. Pseudocode of the algorithm is given in Algorithm 1. The difference between the incremental subgradient method and the vanilla subgradient method is that each agent only computes a subgradient with respect to its own part of the objective function and not the global objective function. The convergence of the incremental subgradient algorithm is guaranteed if the agents can be organized into a cycle graph, which we formalize in the following assumption.

A3: The agents can be organized into a cycle graph. Neighboring nodes in this graph can communicate with each other.

The following proposition guarantees convergence.

Proposition 3 *Under the assumptions A1-A3, Algorithm 1 converges to an optimizer of problem* (4).

PROOF. The proof follows from Theorem 8.2.6 (p. 480) and Theorem 8.2.13 (p. 496) in Bertsekas et al. (2003), since the set Θ is convex and compact (so the norms of all subgradients have an upper bound), and the stepsize α_h is square summable over h, but not summable over h. $\square$

Algorithm 1 can easily be modified to a randomized version (Bertsekas et al., 2003), which corresponds to that the estimate is sent to a random agent at each update. Regardless if the deterministic or the randomized version of the algorithm is used, the convergence behavior is asymptotic, which means that some stopping criteria need to be used. The simplest and most practical criteria is to negotiate for a fixed number of iterations. More advanced stopping criteria are of course possible, but these are outside the scope of this paper.

4 Control Strategies and Numerical Examples

In this section we discuss control strategies and possible extensions. The simplest way to devise a control strategy from the problem formulation is to first execute a *negotiation phase* in which Algorithm 1 is run until a sufficiently accurate optimal consensus point is agreed upon and then, in a *motion phase*, apply the corresponding open-loop control to reach it. If there are no disturbances the system will reach the consensus point at time T. The main advantage of the proposed strategy is that the optimal consensus point is computed in a distributed way and only a small amount of information, the current consensus point estimate, needs to be exchanged at each step. To make the strategy robust to noise, the motion phase can be performed using closed-loop feedback control with the optimal consensus point as a set point. The controller could be devised using, for example, MPC techniques.

Algorithm 1 Cyclic Incremental Algorithm

```
Initialize θ_0 and α_0. Set k := 0 and h := 1.
loop
    α_h := α_0/h
    for i = 1 to N do
        Compute a subgradient, λ_{i,k}, for q_i(θ_k)
        θ_{k+1} := P_Θ{θ_k − α_h λ_{i,k}}
        k := k + 1
    end for
    h := h + 1
end loop
```

We explore the performance of the distributed negotiation. The setup is as follows: three agents with double integrator dynamics and input constraints ($|u_i| \leq 1$) should reach the same coordinates at time $T = 40$. The convergence rate of the consensus point negotiation is shown in Figure 1. The iterations can clearly be seen to converge, and the convergence rate is high in the beginning but slows down after a while. This behavior is typical for algorithms with diminishing stepsizes. For comparison, we solve problem (4) with θ fixed to the mean of the initial positions of the three agents, $\bar{\theta}$. The optimal cost is $\sum_{i=1}^{N} q_i(\theta^\star) = 6446$ and the cost for meeting at $\bar{\theta}$ is 6982. The corresponding optimal trajectories and control signals for agent 2 are shown in Figure 2 and Figure 3, respectively.

5 Discussion

Primal decomposition and incremental subgradient methods provide an interesting framework to pose distributed consensus problems. It allows us to consider general linear models for the agents and easily handle convex input constraints and linear state constraints. The convergence is mathematically guaranteed in the simplest case when negotiation and motion phases are separated. Future work includes the extension of the results to strategies with interleaved negotiation and motion phases and realistic models of the communication network.

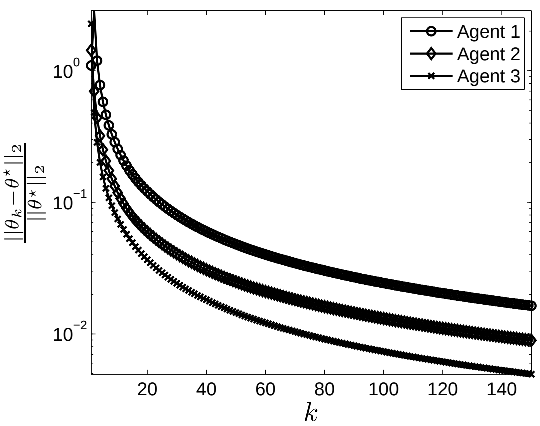

Fig. 1. The consensus point estimates for each agent. The estimates are converging to $\theta^\star$, an optimizer of problem (4).

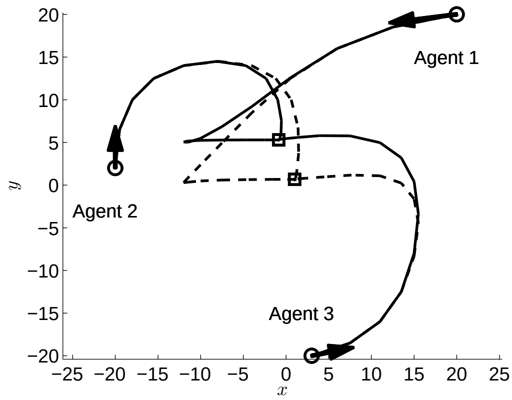

Fig. 2. The trajectories of three agents with double integrator dynamics. The solid lines correspond to the optimal case, $\theta^\star$, and the dashed lines correspond to the mean case, $\bar{\theta}$. The circles are the starting points, the squares are the ending points, and the arrows show the initial velocities.

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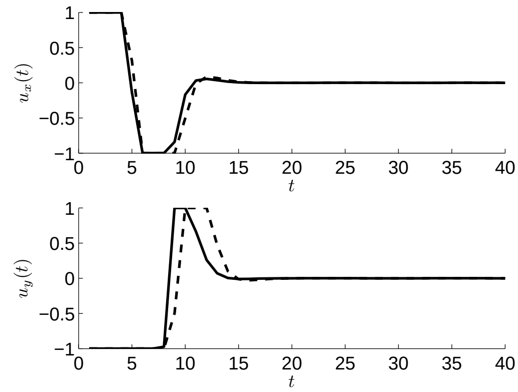

Fig. 3. The control signals for agent 2, where $u_x(t)$ and $u_y(t)$ correspond to the acceleration in the x and y directions, respectively. The solid lines correspond to the optimal case and the dashed lines correspond to the mean case.

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