Symmetries in the Coordinated Consensus Problem *

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Summary. In this paper we consider a widely studied problem in the robotics and control communities, called consensus problem. The aim of the paper is to characterize the relationship between the amount of information exchanged by the vehicles and the speed of convergence to the consensus. Time-invariant communication graphs that exhibit particular symmetries are shown to yield slow convergence if the amount of information exchanged does not scale with the number of vehicles. On the other hand, we show that retaining symmetries in time-varying communication networks allows to increase the speed of convergence even in the presence of limited information exchange.

1 Introduction

The design of coordination algorithms for multiple autonomous vehicles has recently attracted large attention in the control and robotics communities. This is mainly motivated by that multi-vehicle systems have application in many areas, such as coordinated flocking of mobile vehicles [25, 26], cooperative control of unmanned air and underwater vehicles [1, 3], multi-vehicle tracking with limited sensor information [20].

Typically the coordinating vehicles need to communicate data in order to execute a task. In particular they may need to agree on the value of certain coordination state variables. One expects that, in order to achieve coordination, the variables shared by the vehicles, converge to a common value, asymptotically. The problem of designing controllers that lead to such

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asymptotic coordination are called *coordinated consensus* problems, see for example [16, 7, 10, 21], and reference therein. The interest in these type of problems is not limited to the field of mobile vehicles coordination but also involves problems of synchronization [24, 19, 18].

The problem that mostly has been studied in the literature is the design of control strategies that lead to consensus when each vehicle shares its information with vehicles inside a neighborhood [16, 25] and the communication network is time-varying [16, 26, 17].

Robustness to communication link failure [6] and the effects of time delays [21] has also been considered recently. The consensus problem with time-invariant communication networks have been studied in [23, 11]. Randomly time-varying networks have also been analyzed in [14].

In this paper we consider the consensus problem from a different perspective. We are interested to characterize the relationship between the amount of information exchanged by the vehicles and the achievable control performance. More precisely, if we model the communication network by a directed graph, in which each arc represents information transmission from one vehicles to another one, we can expect that good control design methods have to yield better performance for graphs that are more connected. In other words, we model the communication effort of each vehicle as the number of other vehicles it communicates with.

In order to formally characterize the trade off between control performance and communication effort we make the following assumption: the graph topology is independent of the relative position of the vehicles, and the vehicles are described by an elementary first order model. The first hypothesis, that could be realistic in networks of coupled oscillators [19], it is certainly less plausible in applications involving mobile vehicles. Nevertheless a clear analysis of such simplified model seems to be a necessary prerequisite in order to better understand more realistic scenarios. The motivation of describing vehicles with an elementary model is that it allows a quite complete and clean analysis of the problem.

The paper is organized as follows. In section 2 we formally define the consensus problem. In particular we restrict to linear state feedbacks. We then introduce an optimal control problem where the cost functional is related to the convergence rate to the barycenter of the initial position of the vehicles. Under some assumptions, described in section 3, it turns out that weighted directed graphs for which the adjacency matrix is doubly stochastic, are communication graphs that guarantee consensus. Such graphs can be interpreted as a Markov chain and the convergence rate can be related to the mixing rate of the chain [2]. The problem turns out to be treatable for a class of time-invariant graphs with symmetries. In section 4 we introduce the class of Cayley graphs defined on finite Abelian groups. Using tools for bounding the mixing time of Markov chains defined on groups [2, 22] and algebraic properties of finite Abelian groups we derive a bound on the convergence rate to the consensus. The bound is a function of the number of vehicles and the incoming arcs in each vertex of the communication graph, that is the total information each vehicle receives. The main result shows that imposing symmetries in the communication graph, and thus in the control structure, keeping bounded the number of incoming arcs in each vertex, makes the convergence slower as the number of vehicles increases. In section 5 we consider random solutions. In these strategies the communication graph is chosen randomly at each time step over a family of graphs with the constraint that the number of incoming arcs in each vertex is constant. A simple mean square analysis, shows that, in this way, we can improve the convergence rate obtained with time-invariant communication graphs. This holds true even if the random choice is restricted to families of graphs with symmetries. A similar analysis has been proposed in [14] where however a different model of randomly time varying communication graph was proposed and less neat results were obtained. In section 6 some computer simulations are reported.

2 Problem formulation

Consider $N > 1$ vehicles whose dynamics are described by the following discrete time state equations

$$x_i^+ = x_i + u_i \qquad i = 1, \dots, N$$

where $x_i \in \mathbb{R}$ is the state and represents the vehicle position, x_i^+ is the updated state and $u_i \in \mathbb{R}$ is the control input. More compactly we can write

$$x^+ = x + u$$

where $x, u \in \mathbb{R}^N$. The goal is to design a feedback control

$$u = Kx, \qquad K \in \mathbb{R}^{N \times N}$$

yielding the consensus, namely a control that asymptotically makes all the states x_i converging to the same value. More precisely, our objective is to obtain a feedback matrix K such that the closed loop system

$$x^+ = (I + K)x\,,$$

for any initial condition $x(0) \in \mathbb{R}^N$, satisfies

$$\lim_{t \to \infty} x(t) = \alpha v$$

where $v := (1, \dots, 1)^T$ and where α is a scalar depending on $x(0)$.

Without further constraints the above problem is not particularly interesting because it admits completely trivial solutions. Indeed, if $I + K$ is asymptotically stable (condition which can be trivially obtained by choosing, for instance, $K = -I$) then the rendezvous problem is clearly solved with $\alpha = 0$ for all $x(0)$. This is however an inefficient solution since, in this way, nonzero initial states having equal components (in which the rendezvous has already occurred) would produce a useless control action driving all the states to zero. In the following we will impose the condition that the subspace generated by the vector v consists of all equilibrium points, and this happens if and only if

$$Kv = 0\,. \tag{1}$$

From now on, when we say that K solves the rendezvous problem, we will assume that condition (1) is verified. It is easy to see that in this way the rendezvous problem is solved if and only if the following three conditions hold:

(A) the only eigenvalue of $I + K$ on the unit circle is 1;
(B) the eigenvalue 1 has algebraic multiplicity one (namely it is a simple root of the characteristic polynomial of $I + K$) and v is its eigenvector;
(C) all the other eigenvalues are strictly inside the unit circle.

It is clear that, in order to achieve this goal, it is necessary that the vehicles exchange their position. If no constraint is imposed on the amount of information exchanged, it is still quite easy to solve the above problem.

In order to describe the information exchange associated to a specific feedback K it is useful to introduce certain graph theoretic ideas. To any feedback K we associate a directed graph $\mathcal{G}_K$ with set of vertices $\{1, \ldots, N\}$ in which we add an arc from j to i whenever in the feedback matrix K the element $K_{ij} \neq 0$, meaning that in the control of x_i it is used the knowledge of x_j. The graph $\mathcal{G}_K$ is said to be the *communication graph* associated with K. Conversely, given any directed graph $\mathcal{G}$ with set of vertices $\{1, \ldots, N\}$, we say that a feedback K is *compatible* with $\mathcal{G}$ if $\mathcal{G}_K$ is a subgraph of $\mathcal{G}$ (we will use the notation $\mathcal{G}_K \subseteq \mathcal{G}$). We will say that the rendezvous problem is solvable on a graph $\mathcal{G}$ if there exists a feedback K compatible with $\mathcal{G}$ and solving the rendezvous problem. From now on we will always assume that $\mathcal{G}$ contains all loops (i, i): each system has access to its own state.

We would like to have a way to measure the performance of a given control scheme achieving rendezvous. The way to quantify this performance is by no means unique. Suppose we have defined a cost functional $\mathcal{R} = \mathcal{R}(K)$ to be minimized. We can then define

$$\mathcal{R}_{\mathcal{G}} = \min\{\mathcal{R}(K) \mid \mathcal{G}_K \subseteq \mathcal{G}\}\,.$$

We expect a meaningful cost functional to be sensitive to the amount information exchanged by the vehicles, in other words we would like $\mathcal{R}_{\mathcal{G}}$ to show certain range of variation among all the possible communication graphs that can be considered.

The simplest performance index is related to the speed of convergence towards the equilibrium point. Let P be any matrix such that $Pv = v$ and its spectrum (set of the eigenvalues) $\sigma(P)$ is contained in the closed disk centered in 0 and having radius 1. Define

$$\rho(P) = \begin{cases} 1 & \text{if } \dim\ker(P-I) > 1 \\ \max\{|\lambda| : \lambda \in \sigma(P) \setminus \{1\}\} & \text{if } \dim\ker(P-I) = 1\,, \end{cases} \tag{2}$$

we can then take $\mathcal{R}(K) = \rho(I+K)$. The index $\mathcal{R}(K)$ describes the exponential rate of convergence of $x(t)$ towards the equilibrium point. However, such index does not show in general the desired sensitivity to the communication constraints. Indeed, if the graph $\mathcal{G}$ is described by $1 \to 2 \to \cdots \to N$, and we consider the controller

$$K = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

which fulfill condition (1), we obtain $\mathcal{R}(K) = 0$. Thus, adding any communication edge will not lower this index. It is clear however that the above feedback have worse performance than others using a richer communication graph. This means that the spectral radius is not sufficient to highlight these differences.

We then need to refine the model. Since we are considering autonomous vehicles it seems reasonable to consider the cost in terms of control effort, as for instance

$$J(K) := \sum_{t=0}^{\infty} ||u(t)||^2\,.$$

Then the performance index consisting on the pair $(\rho(I+K), J(K))$ would better describe the problem. However this cost is hard to be analyzed. Therefore we consider a simpler index which is related to the previous one, namely

$$J'(K) := \left|\left|\sum_{t=0}^{\infty} u(t)\right|\right|^2 \leq \sum_{t=0}^{\infty} ||u(t)||^2\,.$$

Notice that, in our case, we have that

$$J' = ||x(\infty) - x(0)||^2$$

and so

$$\arg\min\{||x(\infty) - x(0)||^2 : x(\infty) = \alpha v\,, \alpha \in \mathbb{R}\} = \left(\frac{1}{N} v^T x(0)\right) v\,.$$

In this paper we will consider as performance index the pair $(\rho(I+K), J'(K))$, which is relevant and treatable.

Notice that all feedback strategies K producing rendezvous points that are the barycenter of the initial positions of the vehicles, namely such that

$$\lim_{t\to\infty} x(t) = \left(\frac{1}{N} v^T x(0)\right) v\,, \tag{3}$$

are all optimal with respect to the index J'. This feedback maps are called consensus controls [21]. When K yields such a behavior, it will be called a *barycentric controller*. It is easy to see that K is a barycentric controller if and only if

$$v^T K = 0\,. \tag{4}$$

Thus if we restrict to barycentric controllers that satisfy (1) then the performance index of interest is $\rho(I + K)$.

Moreover if we consider the displacement from the barycenter

$$\Delta(t) = x(t) - \left(\frac{1}{N} v^T x(0)\right) v\,,$$

it is immediate to check that, $\Delta(t)$ satisfy the same closed loop equation than $x(t)$. In fact we have

$$\Delta(t+1) = (I + K)\Delta(t)\,. \tag{5}$$

Notice that the initial condition satisfies

$$< \Delta(0), v >= 0\,. \tag{6}$$

Hence the asymptotic behavior of our rendezvous control problem can equivalently be studied by looking at the evolution (5) on the hyperplane characterized by the condition (6). The index $\rho(I + K)$ seems, in this context, appropriate for analyzing the performance.

3 Stochastic and doubly stochastic matrices

If we restrict to control laws K making $I + K$ a nonnegative matrix, condition (1) imposes that $I+K$ is a stochastic matrix. Since the spectral structure of such matrices is quite well known, this observation allows to understand easily what are the conditions on the graph that will ensure the solvability of the rendezvous problem. To exploit this we need to recall some notation and results on directed graphs (the reader can further refer to textbooks on graph theory such as [13]).

Fix a directed graph $\mathcal{G}$ with set of vertices V and set of arcs $\mathcal{E} \subseteq V \times V$. The adjacency matrix A is a $\{0,1\}$ valued square matrix indexed by the elements in V defined by $(A)_{ij} = 1$ if and only $(i,j) \in \mathcal{E}$. Define moreover the in-degree of a a vertex i as $\deg(i) = \sum_j (A)_{ji}$. Vertices of in-degree equal to 0 are called sinks. A graph is called in-regular (of degree k) if each vertex has in-degree equal to k. A path in $\mathcal{G}$ consists of a sequences of vertices $i_1 i_2 \ldots\ldots i_r$ such that $(i_\ell, i_{\ell+1}) \in \mathcal{E}$ for every $\ell = 1, \ldots, r-1$; i_1 (resp. i_r) is said to be the initial (resp. terminal) vertex of the path. A vertex i is said to be connected to a vertex j if there exists a path with initial vertex i and terminal vertex j. A directed graph is said to be connected if given any pair of vertices i and j, at least one of the two is connected to the other. A directed graph is said to be irreducible if given any pair of vertices i and j, i is connected to j. Given any directed graph $\mathcal{G}$ we can consider its irreducible components, namely strongly connected subgraphs $\mathcal{G}_k$ with set of vertices $V_k \subseteq V$ (for $k = 1, \ldots, s$) and set of arcs $\mathcal{E}_k = \mathcal{E} \cap (V_k \times V_k)$ such that the sets V_k form a partition of V. The various components may have connections among each other. We define another directed graph $T_{\mathcal{G}}$ with set of vertices $\{1, \ldots, s\}$ such that there is an arc from k_1 to k_2 if there is an arc in $\mathcal{G}$ from a vertex in V_{k_1} to a vertex in V_{k_2}. It can easily be shown that $T_{\mathcal{G}}$ is always a union of disjoint trees.

Standard results on stochastic matrices [12, page 88 and 99] yield the following proposition.

Proposition 1. *Let $\mathcal{G}$ be a directed graph. The following conditions are equivalent:*

(i) The rendezvous problem is solvable on $\mathcal{G}$.
(ii) $T_{\mathcal{G}}$ is connected and has only one sink vertex.

Moreover if the above conditions are satisfied, any K such that $I + K$ is stochastic, $\mathcal{G}_K = \mathcal{G}$ and $K_{ii} \neq -1$ for every $i \in V_{\mathcal{G}}$ is a possible solution.

Among all possible solutions of the rendezvous problem, when the graph $\mathcal{G}$ satisfies the properties of Proposition 1, a particularly simple one can be written in terms of the adjacency matrix A of $\mathcal{G}$. Consider indeed the matrix P:

$$P_{ij} = \begin{cases} \frac{(A)_{ji}}{\deg(i)} & \text{if } \deg(i) > 0 \\ & \\ 0 & \text{if } \deg(i) = 0 \end{cases}$$

then $K = P - I$ solves the rendezvous problem. Notice the explicit form that the closed loop system assumes in this case:

$$x_i^+ = x_i + \frac{1}{\deg(i)} \sum_{\substack{j \neq i \\ (j,i) \in E}} (x_j - x_i)\,. \tag{7}$$

If we restrict to control laws K making $I+K$ a nonnegative matrix, conditions (1) and (4) are equivalent to the fact that $I+K$ is doubly stochastic. This remark permits to obtain the following result (see [12]).

Proposition 2. *Let $\mathcal{G}$ be a directed graph. The following conditions are equivalent:*

(i) The rendezvous problem on $\mathcal{G}$ admits a barycentric controller.
(ii) $\mathcal{G}$ is irreducible

Moreover if the above conditions are satisfied, any K such that $I+K$ is doubly stochastic, $\mathcal{G}_K = \mathcal{G}$ and $K_{ii} \neq -1$ for every $i \in V$ is a possible solution.

Notice that in the special case when the graph $\mathcal{G}$ is undirected, namely $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$, it follows that we can find solutions K to the rendezvous problem that are symmetric and that therefore are automatically doubly stochastic. One example is given by (7).

We expect the spectral radius to be a meaningful cost functional when restricted to feedback controllers K such that $I+K$ is doubly stochastic. More precisely we conjecture that, by taking

$$\rho_{\mathcal{G}}^{\mathrm{ds}} = \min\{\rho(K) \mid K \text{ doubly stochastic, } \mathcal{G}_K \subseteq \mathcal{G}\}\,,$$

$\mathcal{G}_1 \subset \mathcal{G}_2$ implies that $\rho_{\mathcal{G}_1} > \rho_{\mathcal{G}_2}$. However we have been not able to prove this so far.

4 Symmetric Controllers

The analysis of the rendezvous problem and the corresponding controller synthesis problem becomes more treatable if we limit our search to graphs $\mathcal{G}$ and matrices K exhibiting symmetries. We will show however that these symmetries yield rather poor performance in terms of convergence rate.

In order to treat symmetries on a graph $\mathcal{G}$ in a general setting, we consider Cayley graphs defined on Abelian groups. Let G be any finite Abelian group of order $|G| = N$, and let S be a subset of G which contains the zero. The Cayley graph $\mathcal{G}(G,S)$ is the directed graph with vertex set G and arc set

$$\mathcal{E}_{\mathcal{G}(G,S)} = \{(g,h) : h - g \in S\}\,.$$

Notice that a Cayley graph is always in-regular: the in-degree of each vertex is equal to $|S|$. Notice also that irreducibility can be checked at an algebraic level: it is equivalent to the fact that the set S generates the group G which means that any element in G can be expressed as a finite sum of (not necessarily distinct) elements in S. If S is such that $-S = S$ we say that S is inverse-closed. In this case the graph obtained is undirected.

A Cayley graph supports (stochastic) matrices; to construct them it is sufficient to start from any function $\pi : G \to \mathbb{R}$ such that $\pi(g) = 0$ if $g \notin S$. Then define P by $P_{gh} = \pi(g - h)$. Such a matrix will be called a Cayley matrix (adapted to the Cayley graph $\mathcal{G}(G, S)$). We will also say that P is the Cayley matrix generated by the function π. To have candidate solutions to our rendezvous problem, we of course need something more. Notice that if it holds $\sum_g \pi(g) = 1$, then P satisfies both relations $Pv = v$ and $v^T P = v^T$. In the special case when π is a probability distribution (i.e. $\pi(g) \geq 0$ for every g) P is thus automatically a doubly stochastic matrix: such matrices P will be called Cayley stochastic matrices and for the rest of this section we will mostly work with them. Among the many possible choices of the probability distribution π, there is one which is particularly simple: $\pi(g) = 1/|S|$ for every $g \in S$. In this case we have that

$$P = \frac{1}{|S|} A \,,$$

where A is the adjacency matrix of the Cayley graph $\mathcal{G}(G, S)$.

Example 1. Let us consider the group $\mathbb{Z}_N$ of integers modulo N and the Cayley graph $\mathcal{G}(\mathbb{Z}_N, S)$ where $S = \{-1, 0, 1\}$. Notice that in this case S is inverse-closed. Consider the uniform probability distribution

$$\pi(0) = \pi(1) = \pi(-1) = 1/3$$

The corresponding Cayley stochastic matrix is given by

$$P = \begin{pmatrix} 1/3 & 1/3 & 0 & 0 & \cdots & 0 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1/3 & 0 & 0 & 0 & \cdots & 0 & 1/3 & 1/3 \end{pmatrix} .$$

Notice that in this case we have two symmetries. The first is that the graph is undirected and the second that the graph is circulant. These symmetries can be seen in the structure of the transition matrix P which, indeed, results both symmetric and circulant.

The idea of considering Cayley graphs and Cayley stochastic matrices on Abelian groups is helpful in order to compute, or at least bound, the cost functional $\rho(P)$ defined in (2). We can indeed consider the minimum $\rho_{\mathcal{G}}^{\text{Cayley}}$ of the spectral radius $\rho(P)$ as P varies among the stochastic Cayley matrices compatible with the given Cayley graph $\mathcal{G}$. It will turn out that $\rho_{\mathcal{G}}^{\text{Cayley}}$ can be evaluated or estimated in many cases and clearly it holds $\rho_{\mathcal{G}}^{\text{Cayley}} \geq \rho_{\mathcal{G}}^{\text{ds}}$.

Before continuing we give some short background notions on groups characters and on harmonic analysis on groups, upon which the main results are built.

4.1 Cayley stochastic matrices on finite Abelian groups

Let G be a finite Abelian group of order N, and let $\mathbb{C}^*$ be the multiplicative group of the nonzero complex numbers. A character on G is a group homomorphism $\chi : G \to \mathbb{C}^*$ ($\chi(g+h) = \chi(g)\chi(h)$ for all $g, h \in G$). Since we have that

$$\chi(g)^N = \chi(Ng) = \chi(0) = 1\,, \qquad \forall g \in G$$

it follows that χ takes values on the N^{th}-roots of unity. The character $\chi_0(g) = 1$ for every $g \in G$ is called the trivial character.

The set of all characters of the group G forms an Abelian group with respect to the pointwise multiplication: it is called the character group and denoted by $\hat{G}$. The trivial character χ_0 is the zero of $\hat{G}$. It can be shown that $\hat{G}$ is isomorphic to G, in particular its cardinality is N. If we consider the vector space $\mathbb{C}^G$ of all functions from G to $\mathbb{C}$ with the canonical Hermitian form

$$< f_1, f_2 >= \frac{1}{N} \sum_{g \in G} f_1(g) \overline{f_2(g)}\,,$$

it follows that $\hat{G}$ is an orthonormal basis of $\mathbb{C}^G$.

The Fourier transform of a function $f : G \to \mathbb{C}$ is defined as

$$\hat{f} : \hat{G} \to \mathbb{C}\,, \;\; \hat{f}(\chi) = \sum_{g \in G} \chi(-g) f(g)\,.$$

Example 2. Consider again the group $\mathbb{Z}_N$. The characters are given by

$$\chi_\ell(j) = e^{i \frac{2\pi}{N} \ell j}\,, \quad j \in \mathbb{Z}_N, \quad \ell = 0, \ldots, N-1\,.$$

The correspondence $\ell \to \chi_\ell$ yields an explicit isomorphism between $\mathbb{Z}_N$ and $\hat{\mathbb{Z}}_N$. Given any function $f : \mathbb{Z}_N \to \mathbb{C}$, its Fourier transform is given by

$$\hat{f}(\chi_\ell) = \sum_{j=0}^{N-1} f(j)\, e^{-i \frac{2\pi}{N} \ell j}\,.$$

The cyclic case is instrumental to study characters for any finite Abelian group. Indeed it is a well known result in algebra [15], which states that any finite Abelian group G is isomorphic to a finite direct sum of cyclic groups. In order to study characters of G we can therefore assume that $G = \mathbb{Z}_{N_1} \oplus \cdots \oplus \mathbb{Z}_{N_r}$. It can be shown [2] that the characters of G are precisely the maps $(g_1, g_2, \ldots, g_r) \mapsto \chi^1(g_1)\chi^2(g_2)\cdots\chi^r(g_r)$ with $\chi^i \in \hat{G}_i$ for $i = 1, \ldots, r$. In other terms, $\hat{G}$ is (isomorphic to) $\hat{\mathbb{Z}}_{N_1} \oplus \cdots \oplus \hat{\mathbb{Z}}_{N_r}$. Fix now a Cayley graph on G and a Cayley matrix P generated by the function $\pi : G \to \mathbb{R}$. The spectral structure of P is very simple. To see this, first notice that P can be interpreted as a linear function from $\mathbb{C}^G$ to itself: simply considering, for $f \in \mathbb{C}^G$, $(Pf)(g) = \sum_h P_{gh} f(h)$. Notice that the trivial character χ_0

corresponds to the vector v having all components equal to 1. For every $\chi \in \hat{G}$, it holds

$$(P\chi)(g) = \sum_{h\in G} P_{gh}\chi(h) = \sum_{h\in G} \pi(g-h)\chi(h) = \sum_{h\in S} \pi(h)\chi(g-h) = \hat{\pi}(\chi)\chi(g)\,.$$

Hence, χ is an eigenfunction of P with eigenvalue $\hat{\pi}(\chi)$. Since the characters form an orthonormal basis it follows that P is diagonalizable and its spectrum is given by

$$\sigma(P) = \{\hat{\pi}(\chi) \mid \chi \in \hat{G}\}\,.$$

We can think of characters as linear functions from $\mathbb{C}$ to $\mathbb{C}^G$:

$$\chi :\, z \mapsto z\chi\,,$$

and their adjoint as linear functionals on $\mathbb{C}^{|G|}$:

$$\chi^* :\, f \mapsto < f, \chi > \,.$$

With this notation, $\chi\chi^*$ is a linear function from $\mathbb{C}^G$ to itself, projecting on the eigenspace generated by χ. In this way, P can be represented as

$$P = \sum_{\chi\in\hat{G}} \hat{\pi}(\chi)\chi\chi^*\,.$$

Conversely, it can easily be shown that given any $\theta : \hat{G} \to \mathbb{C}$ the matrix

$$P = \sum_{\chi\in\hat{G}} \theta(\chi)\chi\chi^*\,,$$

is a Cayley matrix generated by the Fourier transform $\pi = \hat{\theta}$.

Suppose now P is the closed loop matrix of a system $x^+ = Px$. The displacement from the barycenter can be represented as

$$\Delta = (I - \chi_0\chi_0^*)x\,.$$

As we had already remarked Δ is governed by the same law (see equation (5))

$$\Delta^+ = P\Delta\,.$$

The initial condition Δ_0 is characterized by $< \Delta_0, \chi_0 >$. Notice that

$$\Delta_t = P^t\Delta_0 = \sum_{\chi\in\hat{G}} \hat{\pi}(\chi)^t\chi < \Delta_0, \chi > \,.$$

Hence,

$$||\Delta_t||^2 = \sum_{\chi\in\hat{G}} |\hat{\pi}(\chi)|^t| < \Delta_0, \chi > |^2\,.$$

This shows in a very simple way, in this case, the role of the spectral radius

$$\rho(P) = \max_{\chi\neq\chi_0} |\hat{\pi}(\chi)|$$

in the convergence performance.

4.2 The spectral radius for stochastic Cayley matrices.

The particular spectral structure of stochastic Cayley matrices allows to obtain asymptotic results on the behavior of the spectral radius $\rho(P)$ and therefore on the speed of convergence of the corresponding control scheme. Let us start with some examples.

Example 3. Consider the group $\mathbb{Z}_N$ and the Cayley graph $\mathcal{G}(\mathbb{Z}_N, S)$, where $S = \{0, 1\}$. Consider the probability distribution π on S described by

$$\pi(0) = 1 - k\,, \;\; \pi(1) = k\,.$$

where $k \in [0, 1]$. The Fourier transform of π is

$$\hat{\pi}(\chi_\ell) = \sum_{g \in S} \chi(-g)\pi(g) = 1 - k + k e^{-i\frac{2\pi}{N}\ell}\,, \qquad \ell = 1, \dots, N-1\,.$$

In this case it can be shown that we have rendezvous stability if and only if $0 < k < 1$ and that the rate of convergence is

$$\rho(P) = \max_{1 \le \ell \le N-1} \left| 1 - k + k e^{-i\frac{2\pi}{N}\ell} \right|\,.$$

Hence, we have that

$$\rho_{\mathcal{G}}^{\text{Cayley}} = \min_k \max_{1 \le \ell \le N-1} \left| 1 - k + k e^{-i\frac{2\pi}{N}\ell} \right|\,.$$

The optimal ℓ and k are $\ell = 1$ and $k = 1/2$ yielding

$$\rho_{\mathcal{G}}^{\text{Cayley}} = \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{N}\right) \right)^{\frac{1}{2}} \simeq 1 - \frac{\pi^2}{2}\frac{1}{N^2}$$

where the last approximation is meant for $N \to \infty$.

Example 4. Consider the group $\mathbb{Z}_N$ and the Cayley graph $\mathcal{G}(\mathbb{Z}_N, S)$, where $S = \{-1, 0, 1, \}$. For the sake of simplicity we assume that N is even; very similar results can be obtained for odd N. Consider the probability distribution π on S described by

$$\pi(0) = k_0\,, \;\; \pi(1) = k_1\,, \;\; \pi(-1) = k_{-1}\,.$$

The Fourier transform of π is in this case given by

$$\hat{\pi}(\chi_\ell) = \sum_{g \in S} \chi(-g)\pi(g) = k_0 + k_1 e^{-i\frac{2\pi}{N}\ell} + k_{-1} e^{i\frac{2\pi}{N}\ell}\,, \qquad \ell = 1, \dots, N-1\,.$$

We thus have

$$\rho_{\mathcal{G}}^{\text{Cayley}} = \min_{(k_0,1,k_{-1})} \max_{1\le \ell \le N-1} \left| k_0 + k_1 e^{-i\frac{2\pi}{N}\ell} + k_{-1} e^{i\frac{2\pi}{N}\ell} \right| .$$

Symmetry and convexity arguments allow to say that a minimum is for sure of the type $k_1 = k_{-1}$. With this assumption the cost functional reduces to

$$\rho(P) = \max_{k_1} \left\{ \left| 1 - 2k_1 \left(1 - \cos\left(\frac{2\pi}{N}\right)\right)\right| , |1 - 4k_1| \right\} .$$

The minimum is achieved for

$$k_0 = \frac{1-\cos\left(\frac{2\pi}{N}\right)}{3-\cos\left(\frac{2\pi}{N}\right)} , \quad k_1 = k_{-1} = \frac{1}{3-\cos\left(\frac{2\pi}{N}\right)}$$

and we have

$$\rho_{\mathcal{G}}^{\text{Cayley}} = \frac{1+\cos\left(\frac{2\pi}{N}\right)}{3-\cos\left(\frac{2\pi}{N}\right)} \simeq 1 - 2\pi^2 \frac{1}{N^2} \tag{8}$$

where the last approximation is meant for $N \to +\infty$.

Notice the asymptotic behavior of previous two examples: the case of communication exchange with two neighbors offer a better performance. However, in both cases $\rho_{\mathcal{G}}^{\text{Cayley}} \to 1$ for $N \to +\infty$. This fact is more general: if we keep bounded the number of incoming arcs in a vertex, the spectral radius for Abelian stochastic Cayley matrices will always converge to 1. This is the content of our main result.

Theorem 1. *Let G be any finite Abelian group of order N and let $S \subseteq G$ be a subset with $|S| = \nu + 1$. Let π be a probability measure associated to the Cayley graph $\mathcal{G}(G, S)$. Then*

$$\rho_{\mathcal{G}(G,S)}^{\text{Cayley}} \geq 1 - CN^{-2/\nu} ,$$

where $C > 0$ is a constant independent of G and S.

Proof. *See Appendix A.*

The consequence of theorem 1 we have that, if we consider any sequence of Cayley stochastic matrices P_N adapted on Abelian Cayley graphs (G_N, S_N) such that $|G_N| = N$ and $|S_N| = o(\ln N)$ then, necessarily, $\rho(P_N)$ converges to 1.

Notice that in Example 4 we have $\nu = 2$ and an asymptotic behavior $\rho_{\mathcal{G}}^{\text{Cayley}} \simeq 1 - 2\pi^2 N^{-2}$ while the lower bound of Theorem 1 is, in this case, $1 - 2\pi^2 N^{-1}$. Can we achieve the bound performance? In other words, is the lower bound we have just found, tight? The following example shows that this is the case.

Example 5. Consider the group $\mathbb{Z}_N^\nu$ and the Cayley graph $\mathcal{G}(\mathbb{Z}_N^\nu, S)$, where $S = \{0, e_1, \ldots, e_\nu\}$ where e_j is the vector with all elements equal to 0 except a 1 in position j. Consider the probability distribution π on S described by

$$\pi(0) = \pi(e_i) = \frac{1}{\nu+1}\,,\ \forall i = 1, \ldots, \nu\,.$$

The Fourier transform of π is

$$\hat{\pi}(\chi_{\ell_1}, \ldots, \chi_{\ell_\nu}) = \sum_{g \in S} \chi(-g)\pi(g) = \frac{1}{\nu+1}\left(1 + \sum_{j=1}^{\nu} e^{-i\frac{2\pi}{N}\ell_j}\right).$$

where $\ell_j = 1, \ldots, N-1\,,\ j = 1, \ldots, \nu$. We thus have that for this graph

$$\rho_{\mathcal{G}} = \frac{1}{\nu+1} \max_{1 \le \ell_j \le N-1} \left|\nu + \sum_{j=1}^{\nu} e^{-i\frac{2\pi}{N}\ell_j}\right|.$$

It is easy to see that the above min-max is reached by $\ell_j = 1$ for every $j = 1, \ldots, \nu$ or when $\exists h \in \{1, \nu\}$ such that $\ell_h = 1$ and for all $j \neq h$ we have $\ell_j = 0$. This yields the value

$$\rho_{\mathcal{G}} = \left(\frac{\nu^2+1}{(\nu+1)^2} + \frac{2\nu}{(\nu+1)^2}\cos\left(\frac{2\pi}{N}\right)\right)^{\frac{1}{2}} \simeq 1 - C\frac{\pi^2}{N^2} = 1 - C\frac{\pi^2}{(N^\nu)^{2/\nu}}$$

where C is a constant. Notice we have exactly obtained the lower bound proven above.

5 Random communication graphs

5.1 Random circulant communication graph

A direct graph $\mathcal{G} = (V, \mathcal{E})$ is said to be a *circulant directed graph* if $(i, j) \in \mathcal{E}$ implies that $(i + p \mod N, j + p \mod N) \in \mathcal{E}$ where $p \in \mathbb{N}$. Observe that in a direct circulant graph each vertex has the same in-degree. In the following sometimes we will refer to the in-degree of the graph, meaning the in-degree of any of the vertices of the graph.

Let $\bar{\mathcal{G}}$ the set of all circulant directed graphs $\mathcal{G} = (V, \mathcal{E})$ with in-degree $\nu+1$ and such that the corresponding adjacency matrix $\mathcal{A}$ has $a_{ii} \neq 0,\ \forall\, 1 \le i \le N$, meaning that, as we said perviously, each vehicle has access to its own state.

In this strategy we suppose that at each time instant t the communication graph $\mathcal{G}(t)$ is chosen randomly from the set $\bar{\mathcal{G}}$ accordingly to a uniform distribution. This is equivalent to impose that the adjacency matrix $\mathcal{A}(t)$ of the communication graph $\mathcal{G}(t)$ is such that

$$\mathcal{A}(t) = I + \sum_{i=1}^{\nu} \Pi^{\alpha_i(t)}$$

where Π is the following circulant matrix [8]

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and where $\{\alpha_1(t)\}, \ldots, \{\alpha_\nu(t)\}$ are ν independent sequences of independent random variables uniformly distributed over the alphabet $\mathcal{X} = \{1, 2, \ldots, N\}$. We consider the following control law $u(t)$

$$u(t) = \left(k_0 I + \sum_{i=1}^{\nu} k_i \Pi^{\alpha_i(t)} \right) x(t). \tag{9}$$

The close loop system then becomes

$$x(t+1) = \left((1+k_0) I + \sum_{i=1}^{\nu} k_i \Pi^{\alpha_i(t)} \right) x(t) \tag{10}$$

The system (10) can be regarded as a Markov jump linear system [4].

Notice that the state transition matrix in (10) is a circulant matrix and since $\Pi v = v$ we have that the conditions (1) is satisfied. If we restrict to non-negative matrices then we have that the feedback gains $k_0, \ldots, k_\nu$ are such that $1 + k_0, k_1, \ldots, k_\nu \geq 0$ and $k_0 + \sum_{i=1}^{\nu} k_i = 0$.

We conclude this section by observing that this strategy has an evident drawback from an implementation point of view : the same random choice, done at every time instance, needs to be known by all vehicles. A possible way to overcome this limitation is by using a predetermined pseudo-random sequence whose starting seed is known to everybody.

5.2 Random communication graph with bounded in-degree

The strategy that we consider in this subsection is similar to the one presented in the previous subsection, but it overcomes the implementation issues. In this case we do not limit the time-varying communication graph $\mathcal{G}(t)$ to be circulant. We assume that each vehicle, besides knowing its own position, receives the state of ν vehicle chosen randomly and independently. Because of this it can happen that the resulting communication graph $\mathcal{G}(t)$ could have multiple arcs connecting the same pair of nodes.

The feedback control in this case is

$$u(t) = k_0 x(t) + \sum_{i=1}^{\nu} k_i E_i(t) x(t) \tag{11}$$

where $\{E_i(t)\}$, $i = 1, \dots, \nu$, are ν independent sequences of independent random processes taking value on the set of matrices

$$\Upsilon := \left\{ E \in \{0,1\}^{N\times N} \ : \ Ev = v \right\}$$

and equally distributed in such a set. Roughly speaking, the set Υ is constituted by all matrices with entries 0 or 1 that have exactly one element equal to 1 in each row. Since $E_i(t)v = v$, for all $i = 1, \dots, \nu$ and all $t \geq 0$, we have that, as in the previous case, the in order for condition (1) to hold and to have a non-negative matric, the feedback gains $k_0, \dots, k_\nu$ must satisfy $1 + k_0, k_1, \dots, k_\nu \geq 0$ and $k_0 + \sum_{i=1}^{\nu} k_i = 0$.

The close loop system becomes

$$x(t+1) = (1 + k_0) I + \sum_{i=1}^{\nu} k_i E_i(t) x(t)\,. \tag{12}$$

Notice that also the system (12) can be regarded as Markov jump linear system.

5.3 Convergence and performance analysis

In order to study the asymptotic behavior of the two previous strategy, it is convenient to introduce the variable $y(t)$ that is defined in the following way. Consider

$$Y = I - \frac{1}{N} v v^T \tag{13}$$

and let

$$y(t) = Y x(t). \tag{14}$$

Notice that the component $y_i(t)$ of $y(t)$, represents, by this definition, the displacement of $x_i(t)$ from the barycenter of the initial position of the vehicles, at the time instant t. Clearly we have that

$$\lim_{t\to+\infty} x(t) = \alpha v \tag{15}$$

if and only if

$$\lim_{t\to+\infty} y(t) = 0\,. \tag{16}$$

Note that it holds both $Y\Pi = \Pi Y$ and $YE = YEY$ where E is any matrix in Υ. Then pre-multiplying (10) and (12) by Y we obtain

$$y(t+1) = \left((1+k_0)I + \sum_{i=1}^{\nu} k_i \Pi^{\alpha_i(t)} \right) y(t) \tag{17}$$

if we consider the first strategy and

$$y(t+1) = F_2(t) = \left((1+k_0)I + \sum_{i=1}^{\nu} k_i E_i(t) \right) y(t) \tag{18}$$

In order to study the asymptotic properties of $y(t)$ we consider $\mathbb{E}\left[\|y(t)\|^2\right]$ where the expectation is taken over the set of the graph in which $\mathcal{G}(t)$ is chosen. We then have the following definition.

Definition 1 ([9]). *The jump linear Markov systems* (17) *and* (18) *are said to be asymptotically second moment stable if for any fixed* $y(0) \in \mathbb{R}^N$ *it holds*

$$\lim_{t \to +\infty} \mathbb{E}\left[\|y(t)\|^2\right] = 0. \tag{19}$$

We can then state the two main results.

Proposition 3. *The system* (17) *is asymptotically second moment stable for any initial condition* $y(0)$ *and for* $k_0 = -\nu/(1+\nu)$ *and* $k_i = 1/(1+\nu)$, $1 \le i \le \nu$. *Moreover, for these values of* k_i *we obtain the fasted convergence rate and we have that*

$$\mathbb{E}\left[\|y(t)\|^2\right] = \left(\frac{1}{1+\nu} \right)^t \|y(0)\|^2. \tag{20}$$

Proof. See Appendix B.

Proposition 4. *The system* (18) *is asymptotically second moment stable for any initial condition* $y(0)$ *and for* $k_0 = -\nu N/(N + N\nu - 1)$ *and* $k_i = N/(N + N\nu - 1)$, $1 \le i \le \nu$. *Moreover, for these values of* k_i *we obtain the fastest convergence rate and we have that*

$$\mathbb{E}\left[\|y(t)\|^2\right] = \left(\frac{N-1}{N(1+\nu)-1} \right)^t \|y(0)\|^2. \tag{21}$$

Proof. See Appendix C.

Remark 1. Notice that the convergence rate obtained using the strategy with random circulant communication graphs it does not depend on N and ensures convergence even if $\nu = 1$, which corresponds to the case when each vehicle receives the state of at most one another vehicle. The strategy with random communication graphs with in-degree bounded, attains the same converge rate of the first only from $N \to +\infty$.

However notice that both strategies have a better convergence rate than the one obtained using time-invariant communication graphs. This increase of performance has obtained by randomizing the choice over a pre-assigned family of graphs.

Remark 2. Notice that by using random communication graphs with bounded in-degree the vehicles, in general, will not reach the consensus at the barycenter of the initial configurations, since

$$v^T \left(k_0 I + \sum_{i=1}^{\nu} k_i E_i(t) \right) \neq v^T \tag{22}$$

It is meaningful to study where the vehicles will reach consensus with respect to the barycenter of the initials conditions. In order to carry out this analysis we consider the mean square distance from the barycenter of the initials conditions namely we consider $x(t) - v^T x(0) v/N$. We have the following result.

Proposition 5. *The mean square distance to the barycenter of the initial configuration of the vehicle is bounded, namely,*

$$\lim_{t\to\infty} \mathbb{E}\left[(x(t) - bv)(x(t) - bv)^T\right] = \alpha \frac{1}{N} x(0)^T (I - \Omega) x(0)$$

where

$$\alpha = \frac{\sum_{i=1}^{\nu} k_i^2}{(1-N)\sum_{1=1}^{\nu} k_i^2 - k_0 N (k_0 + 2)}$$

and where $b = v^T x(0)/N$, *and* $\Omega = vv^T/N$ *and where the* k_i *make the system* (18) *asymptotically second order stable.*

Proof. See Appendix D.

Notice that when $N \to +\infty$ then the mean square distance to the barycenter tends to zero. If we use the control gains $k_0 = -\frac{\nu N}{N + N\nu - 1}$ and $k_1 = \cdots = k_\nu = N/(N + N\nu - 1)$ then we have that

$$\lim_{t\to\infty} \mathbb{E}\left[(x(t) - bv)(x(t) - bv)^T\right] = \frac{1}{N(N(1+\nu) - 1)} x(0)^T (I - \Omega) x(0) .$$

Notice that for fixed N if ν grows the mean square distance to the barycenter becomes smaller.

6 Simulation results

In Figures 1-3, some computer simulation results are reported. The simulations show the time evolution of the state of $n = 10$ vehicles when they can exchange the state with at most one other vehicle, thus in this case we have $\nu = 1$. The initial condition for all the three simulations is the same and the barycenter coordinate is 2.76. Figure 1 refers to the state evolution for time-invariant communication graph when the Cayley graph is the ring graph. Figure 2 refers to random circulant communication graph and Figure 3 refers to random communication graph with bounded in-degree. Notice that the random strategies, as expected, exhibit a faster rate convergence and that the random communication graph with bounded in-degree does not converge to the barycenter (in this case the consensus is reached at 0.71).

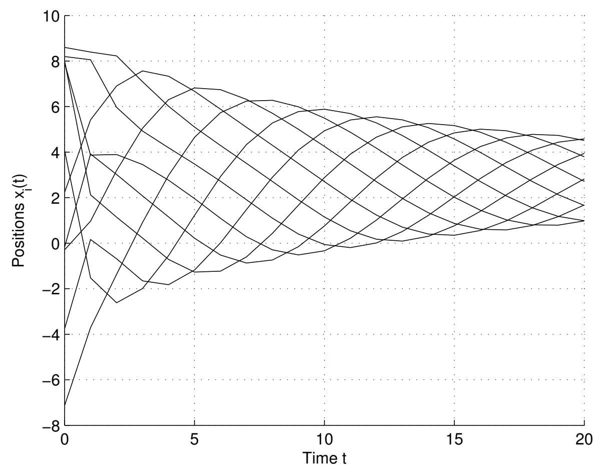

Fig. 1. Time-invariant Cayley graph with $\nu = 1$.

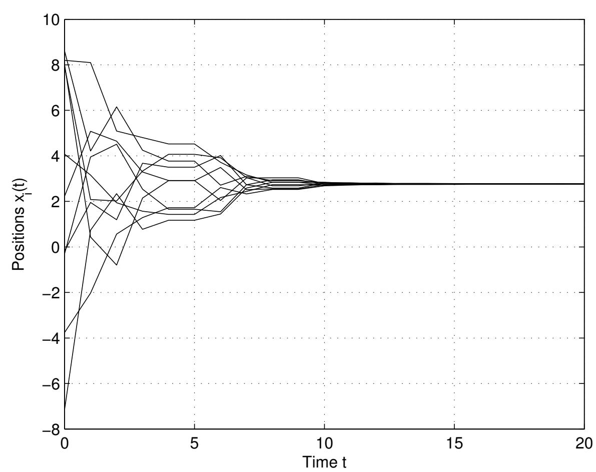

Fig. 2. Random circulant communication graph.

7 Conclusions

In this paper we have analyzed the relationship between the communication graph and the convergence rate to the rendezvous point for a team of vehicles. Modelling the communication graph with a Cayley graph defined on Abelian groups, namely a graph with symmetries, we have been able to bound the convergence rate. In particular we have proved that the convergence to the barycenter of the initial configuration becomes slower and slower as the num-

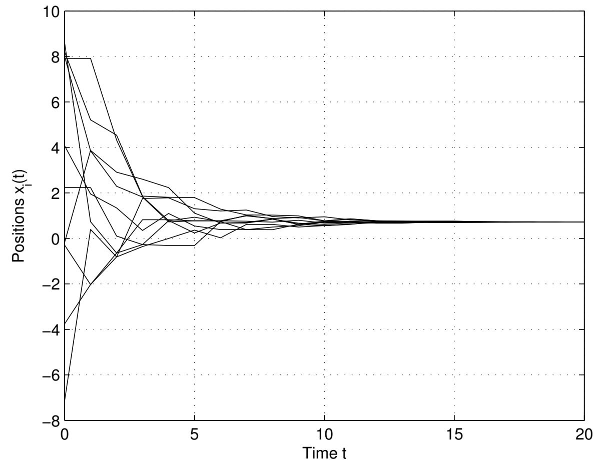

Fig. 3. Random communication graph with bounded in-degree.

ber of vehicles increases if the amount of information received by each vehicle remain constant. We have also considered some particular random strategies that consist in choosing randomly a communication graph in a predefined family of graphs. In particular we have considered the circulant graphs and graphs with bounded in-degree. It turns out that choosing randomly over such family graphs we obtain higher performances then using time-invariant communication graphs. In [5] the analysis has been extended to random Cayley graphs and to communication graphs where the information is quantized.

A Proof of Theorem 1

In order to prove theorem 1 we need the following technical lemma.

Lemma 1. *Let* $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong [-1/2, 1/2[$. *Let* $0 \leq \delta \leq 1/2$ *and consider the hypercube* $V = [-\delta, \delta]^k \subseteq \mathbb{T}^k$. *For every* $\Lambda \subseteq \mathbb{T}^k$ *such that* $|\Lambda| \geq \delta^{-k}$, *there exist* $\bar{x}_1, \bar{x}_2 \in \Lambda$ *with* $\bar{x}_1 \neq \bar{x}_2$ *such that* $\bar{x}_1 - \bar{x}_2 \in V$.

Proof. For any $x \in \mathbb{T}$ and $\delta > 0$, define the following set

$$L(x, \delta) = [x, x + \delta] + \mathbb{Z} \subseteq \mathbb{T}\,.$$

Observe that for all $y \in \mathbb{T}$, $L(x, \delta) + y = L(x+y, \delta)$. Now let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_k) \in \mathbb{T}^k$ and define

$$L(\bar{x}, \delta) = \prod_{i=1}^{k} L(\bar{x}_i, \delta)\,.$$

Also in this case we observe that $L(\bar{x},\delta)+\bar{y} = L(\bar{x}+\bar{y},\delta)$ for every $\bar{y} \in \mathbb{T}^k$. Consider now the family of subsets

$$\{L(\bar{x},\delta), \;\; \bar{x} \in \Lambda\}\,.$$

We claim that there exist $\bar{x}_1$ and $\bar{x}_2$ in Λ such that $\bar{x}_1 \neq \bar{x}_2$ and such that $L(\bar{x}_1,\delta) \cap L(\bar{x}_2,\delta) \neq \emptyset$. Indeed, if not, we would have $|\Lambda|\delta^k < 1$ which contradicts our assumptions. Notice finally that

$$L(\bar{x}_1,\delta) \cap L(\bar{x}_2,\delta) \neq \emptyset \;\Leftrightarrow\; L(0,\delta) \cap L(\bar{x}_2-\bar{x}_1,\delta) \neq \emptyset \;\Leftrightarrow\; \bar{x}_2-\bar{x}_1 \in V\,.$$

■

We can now prove theorem 1.

Proof. With no loss of generality we can assume that

$$G = \mathbb{Z}_{N_1} \oplus \ldots \oplus \mathbb{Z}_{N_r}\,.$$

Assume we have fixed a probability distribution π concentrated on S. Let P be the corresponding stochastic Cayley matrix. It follows from previous considerations that the spectrum of P is given by

$$\sigma(P) = \left\{ \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_r=0}^{N_r-1} \pi(k_1,\ldots,k_r) e^{i\frac{2\pi}{N_1}k_1\ell_1} \cdots e^{i\frac{2\pi}{N_r}k_r\ell_r} : \ell_s \in \mathbb{Z}_{N_s} \; \forall s = 1,\ldots,r \right\}$$

Denote by $\bar{k}^j = (k_1^j,\ldots,k_r^j)$, for $j = 1,\ldots,\nu$, the non-zero elements in S, and consider the subset

$$\Lambda = \left\{ \left(\sum_{i=1}^r \frac{k_i^1 \ell_i}{N_i}, \ldots, \sum_{i=1}^r \frac{k_i^\nu \ell_i}{N_i} \right) + \mathbb{Z}^k \mid \ell_s \in \mathbb{Z}_{N_s} \text{ for } 1 \leq s \leq r \right\} \subseteq \mathbb{T}^\nu\,.$$

Let $\delta = (\prod_i N_i)^{-1/\nu}$ and let V be the corresponding hypercube defined as in Lemma 1. We claim that there exists $\bar{\ell} = (\ell_1,\ldots \ell_r) \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$, $\bar{\ell} \neq 0$ such that

$$\left(\sum_{i=1}^r \frac{k_i^1 \ell_i}{N_i}, \ldots, \sum_{i=1}^r \frac{k_i^\nu \ell_i}{N_i} \right) + \mathbb{Z}^k \in V\,.$$

Indeed, if there exist two different $\bar{\ell}', \bar{\ell}'' \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$ such that

$$\left(\sum_{i=1}^r \frac{k_i^1 \ell_i'}{N_i}, \ldots, \sum_{i=1}^r \frac{k_i^\nu \ell_i'}{N_i} \right) + \mathbb{Z}^\nu = \left(\sum_{i=1}^r \frac{k_i^1 \ell_i''}{N_i}, \ldots, \sum_{i=1}^r \frac{k_i^\nu \ell_i''}{N_i} \right) + \mathbb{Z}^\nu\,,$$

then we have that,

$$\left(\sum_{i=1}^r \frac{k_i^1 \ell_i}{N_i}, \ldots, \sum_{i=1}^r \frac{k_i^\nu \ell_i}{N_i} \right) + \mathbb{Z}^\nu = 0\,,$$

where $\bar{\ell} = \bar{\ell}' - \bar{\ell}'' \neq 0$. On the other hand, if different elements in $\mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$ always lead do distinct elements in Λ, then, $|\Lambda| = \prod_i N_i = \delta^{-\nu}$. We can then apply Lemma 1 and conclude that there exist two different $\bar{\ell}', \bar{\ell}'' \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$ such that

$$\left[\left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i'}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i'}{N_i}\right) + \mathbb{Z}^\nu\right] - \left[\left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i''}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i''}{N_i}\right) + \mathbb{Z}^\nu\right] \in V\,.$$

Hence,

$$\left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i}{N_i}\right) + \mathbb{Z}^\nu \in V\,,$$

where $\bar{\ell} = \bar{\ell}' - \bar{\ell}'' \neq 0$. Consider now the eigenvalue

$$\begin{aligned}\lambda &= \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \cdots \sum_{k_r=0}^{N_r-1} \pi(k_1, \ldots, k_r) e^{i(\frac{2\pi}{N_1} k_1 \ell_1 + \frac{2\pi}{N_2} k_2 \ell_2 + \cdots + \frac{2\pi}{N_r} k_r \ell_r)} \\ &= \pi(0, \ldots 0) + \sum_{j=1}^{\nu} \pi(k_1^j, \ldots, k_r^j) e^{j(\frac{2\pi}{N_1} k_1^j \ell_1 + \frac{2\pi}{N_2} k_2^j \ell_2 + \cdots + \frac{2\pi}{N_r} k_r^j \ell_r)}\,.\end{aligned}$$

Its norm can be estimated as follows

$$\begin{aligned}&|\lambda| \geq \pi(0, \ldots 0) + \sum_{j=1}^{\nu} \pi(k_1^j, \ldots, k_r^j) \cos\left(\frac{2\pi}{N_1} k_1^j \ell_1 + \frac{2\pi}{N_2} k_2^j \ell_2 + \cdots + \frac{2\pi}{N_r} k_r^j \ell_r\right) \\ &\geq \pi(0, \ldots 0) + \sum_{j=1}^{\nu} \pi(k_1^j, \ldots, k_r^j) \; - \sum_{j=1}^{\nu} \pi(k_1^j, \ldots, k_r^j) 2\pi^2 \frac{1}{N^{2/\nu}} \geq 1 - 2\pi^2 \frac{1}{N^{2/\nu}}\end{aligned}$$

and so we can conclude. ■

B Proof of Proposition 1.

As previously we observe that $\mathbb{E}\left[\|y(t)\|\right]^2 = \operatorname{tr} \mathbb{E}\left[y(t) y^T(t)\right]$. Let $P(t) = \mathbb{E}[y(t) y^T(t)]$. We have that

$$\begin{aligned}P(t+1) &= \mathbb{E}\left[y(t+1) y^T(t+1)\right] \\ &= \mathbb{E}\left[\left((1+k_0)I + \sum_{i=1}^{\nu} \Pi^{\alpha_i(t)}\right) y(t) y^T(t) \left((1+k_0)I + \sum_{i=1}^{\nu} \Pi^{\alpha_i(t)}\right)^T\right] \\ &= \mathbb{E}\Bigg[\mathbb{E}\Bigg[\left((1+k_0)I + \sum_{i=1}^{\nu} \Pi^{\alpha_i(t)}\right) y(t) \times \\ &\qquad \times y^T(t) \left((1+k_0)I + \sum_{i=1}^{\nu} \Pi^{\alpha_i(t)}\right)^T \mid \alpha_1(t), \ldots, \alpha_\nu(t)\Bigg]\Bigg]\end{aligned}$$

Since $y(t)$ is independent from $\alpha_1(t), \dots, \alpha_\nu(t)$ we obtain

$$\begin{aligned}
P(t+1) = \mathbb{E}&\left[\left((1+k_0)I + \sum_{i=1}^{\nu} \Pi^{\alpha_i(t)}\right) P(t) \left((1+k_0)I + \sum_{i=1}^{\nu} \Pi^{\alpha_i(t)}\right)^T\right] \\
&= (1+k_0)^2 P(t) + (1+k_0)\big(\sum_{i=1}^{\nu} k_i \mathbb{E}(\Pi^{\alpha_i(t)})\big)P(t) + \\
&\quad + (1+k_0)\big(\sum_{i=1}^{\nu} k_i \mathbb{E}(\Pi^{-\alpha_i(t)})\big)P(t) + \\
&\quad + \sum_{i=1}^{\nu} \sum_{\substack{j=1 \\ j \neq i}}^{\nu} k_i k_j \mathbb{E}\left[\Pi^{\alpha_i(t)}\right] P(t) \mathbb{E}\left[\Pi^{-\alpha_j(t)}\right] + \\
&\quad + \sum_{i=1}^{\nu} k_i^2 \mathbb{E}\left[\Pi^{\alpha_i(t)} P(t) \Pi^{-\alpha_i(t)}\right]
\end{aligned}$$

By straightforward calculations one can verify that $\mathbb{E}\left[\Pi^{-\alpha(t)}\right] = \mathbb{E}\left[\Pi^{\alpha(t)}\right] = \frac{1}{N} v v^T$ and $\operatorname{tr}\left(P(t) v v^T\right) = 0$. Hence

$$\mathbb{E}\left[\|y(t+1)\|^2\right] = (1+k_0)^2 \mathbb{E}\left[\|y(t)\|^2\right] + \sum_{i=1}^{\nu} k_i^2 \operatorname{tr} \mathbb{E}\left[\Pi^{\alpha_i(t)} P(t) \Pi^{-\alpha_i(t)}\right] \quad (23)$$

Using the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ we can conclude that

$$\begin{aligned}
\mathbb{E}\left[\|y(t+1)\|^2\right] &= \left((1+k_0)^2 + \sum_{i=1}^{\nu} k_i^2\right) \mathbb{E}\left[\|y(t)\|^2\right] \\
&= \left((1+k_0)^2 + \sum_{i=1}^{\nu} k_i^2\right)^t \|y(0)\|^2
\end{aligned}$$

Now it is easy to verify that

$$\min\left\{(1+k_0)^2 + \sum_{i=1}^{\nu} k_i^2 \mid 1+k_0, k_1, \dots, k_\nu \geq 0, \sum_{j=0}^{\nu} k_j = 0\right\} = \frac{1}{1+\nu} \quad (24)$$

and that it is obtained by choosing $k_0 = -\nu/(1+\nu)$ and $k_j = 1/(\nu+1)$ for all $1 \leq j \leq \nu$. With such a choice we have the convergence result (20)

■

C Proof of Proposition 2.

We observe that $\mathbb{E}\left[\|y(t)\|\right]^2 = \operatorname{tr} \mathbb{E}\left[y(t) y^T(t)\right]$. Let $P(t) = \mathbb{E}[y(t) y^T(t)]$. We have that

$$\begin{aligned}P(t+1) = \mathbb{E}\Bigg[\Bigg((1+k_0)y(t) + \sum_{i=1}^{\nu} k_i Y E_i(t) y(t)\Bigg) \times \\ \times \Bigg((1+k_0)y(t) + \sum_{i=1}^{\nu} k_i Y E_i(t) y(t)\Bigg)^T\Bigg] \\ = (1+k_0)^2 P(t) + \mathbb{E}\left[(1+k_0)y(t)y^T(t)\sum_{i=1}^{\nu} k_i E_i^T(t) Y\right] + \\ + \mathbb{E}\left[\sum_{i=1}^{\nu} Y E_i(t) y(t)\, y^T(t)(1+k_0)\right] + \\ + \; E\left[\sum_{i=1}^{\nu} k_i Y E_i(t) y(t) y^T(t) \sum_{j=1}^{\nu} k_j E_j^T(t) Y\right].\end{aligned}$$

Using the double expectation theorem and the fact that $E_i(t)$ and $P(t)$ are independent for any $i = 1, \dots, r$, we have that

$$\begin{aligned}P(t+1) = (1+k_0)^2 P(t) + (1+k_0)P(t)\,\mathbb{E}\left[\sum_{i=1}^{\nu} k_i E_i^T(t) Y\right] + \\ + (1+k_0)\,\mathbb{E}\left[\sum_{i=1}^{\nu} k_i Y E_i(t)\right] P(t) + \\ + Y\,\mathbb{E}\left[\sum_{i=1}^{\nu}(k_i E_i(t)) P(t) \sum_{j=1}^{\nu}(k_j E_j^T(t))\right] Y.\end{aligned}$$

Let $\Omega = vv^T/N$. Since $\mathbb{E}[E_i(t)] = \Omega$ and since $Y\,\Omega = \Omega\,Y = 0$ we have that the first two expectations in the previous equation are equal to zero. To compute the last expectation we need to distinguish two cases:

$i \neq j$: then $E_i(t)$, $E_j^T(t)$ and $P(t)$ are all independent and thus the expectation factorizes. Two terms of the type $Y\,\Omega$ appear and thus for $i \neq j$ the expectation is zero,

$i = j$: then, since it can be verified by straightforward calculations that for any $M \in \mathbb{R}^{N\times N}$,

$$\mathbb{E}\left[E_i(t) M E_i^T(t)\right] = \frac{1}{N}(v^T M v)\Omega + \left(\frac{1}{N}\operatorname{tr} M - \frac{1}{N^2} v^t M v\right) I,$$

we have

$$
\begin{aligned}
Y\mathbb{E}\left[k_i E_i(t)\, P(t)\, k_i E_i^T(t)\right] Y &= k_i^2 Y\mathbb{E}\left[E_i(t)P(t)E_i^T(t)\right] Y \\
&= \frac{k_i^2}{N} Y v^T P(t)\, v\, \Omega Y + \frac{k_i^2}{N}\operatorname{tr}\left(P(t)\right) Y \\
&\quad - \frac{k_i^2}{N^2} Y v^T P(t)\, vY
\end{aligned}
$$

The first term of the previous equation is zero since ΩY is zero.

We thus obtain that

$$
P(t+1) = (1+k_0)^2 P(t) + \frac{1}{N}\sum_{i=1}^{\nu} k_i^2 \operatorname{tr}\left(P(t)\right)Y - \frac{1}{N^2}\sum_{i=1}^{\nu} k_i^2 Y v^T\, P(t)\, vY.
$$

Now we consider $\mathbb{E}\left[\|y(t)\|^2\right] = \operatorname{tr} P(t)$, then we have

$$
\begin{aligned}
\mathbb{E}\left[\|y(t+1)\|^2\right] = (1+k_0)^2\mathbb{E}\left[\|y(t)\|^2\right] &+ \frac{1}{N}\sum_{i=1}^{\nu} k_i^2 \operatorname{tr}\left(\operatorname{tr}\left(P(t)\right)Y\right) \\
&- \frac{1}{N^2}\sum_{i=1}^{\nu} k_i^2 \operatorname{tr}\left(v^T\, P(t)\, vY\right).
\end{aligned}
$$

The term $\operatorname{tr}\left(\operatorname{tr}\left(P(t)\right)Y\right) = (N-1)\operatorname{tr}\left(P(t)\right)$ since $\operatorname{tr}\left(Y\right) = N-1$ and the last term is zero since

$$
v^T\, P(t)\, v = \sum_{i=1}^{N}\sum_{j=1}^{N}(P(t))_{ij} = 0\,.
$$

We thus have the following difference equation

$$
\mathbb{E}\left[\|y(t+1)\|^2\right] = \left((1+k_0)^2 + \frac{N-1}{N}\sum_{i=1}^{\nu} k_i^2\right)\mathbb{E}\left[\|y(t)\|^2\right]\ .
$$

Now it is easy to verify that

$$
\min_{k,k_1,\ldots,k_\nu}\left\{(1+k_0)^2 + \frac{N-1}{N}\sum_{i=1}^{\nu} k_i^2 \mid 1+k_0, k_1,\ldots,k_\nu \geq 0, \sum_{j=1}^{\nu} k_j = 1\right\} = \frac{1}{1+\nu} \tag{25}
$$

and that it is obtained by choosing $k_0 = -\nu N/(N(1+\nu)-1)$ and $k_j = \nu N/(N(1+\nu))$ for all $1 \leq j \leq \nu$. With such a choice we have the convergence result (21). ■

D Proof of Proposition 3.

Let us define $z(t) = x(t) - b\,v$. It is not difficult to prove that $z(t)$ has the same close loop dynamics as the system in $x(t)$, thus

$$z(t+1) = \left((1+k_0)I + \sum_{i=1}^{\nu} k_i E_i \right) z(t).$$

Let us consider $P(t) = \mathbb{E}[(x(t) - bv)(x(t) - bv)^T] = \mathbb{E}[z(t)z^T(t)]$. Similar calculations as those done for the convergence rat yields

$$\begin{aligned} P(t+1) = (1+k_0)^2 P(t) + (1+k_0) \sum_{i=1}^{\nu} k_i \left(P(t)\Omega + \Omega P(t) \right) + \\ + \sum_{i=1}^{\nu} k_i^2 \left(\frac{1}{N}(v^T P(t)\, v)\Omega + \left(\frac{1}{N}\operatorname{tr} P(t) - \frac{1}{N^2} v^T P(t)\, v \right) I \right) + \\ + \sum_{\substack{i=1 \\ i \neq j}}^{\nu} \sum_{j=1}^{\nu} k_i k_j \, \Omega P(t) \Omega \end{aligned}$$

Let us define the following variables

$$\begin{aligned} w(t) &= \frac{1}{N}\operatorname{tr} P(t) \\ s(t) &= \frac{1}{N^2}\, v^T P(t) v \end{aligned}$$

We want to compute the mean squared distance from the barycenter at steady state, namely we want to compute $w(\infty) := \lim_{t\to\infty} w(t)$. We have that

$$\begin{pmatrix} w(t+1) \\ s(t+1) \end{pmatrix} = \underbrace{\begin{pmatrix} (1+k_0)^2 + \sum_{i=1}^{\nu} k_i^2 & -k_0(k_0+2) - \sum_{i=1}^{\nu} k_i^2 \\ \frac{1}{N}\sum_{i=1}^{\nu} k_i^2 & 1 - \frac{1}{N}\sum_{i=1}^{\nu} k_i^2 \end{pmatrix}}_{\Sigma} \begin{pmatrix} w(t) \\ s(t) \end{pmatrix} .$$

where the transition matrix Σ has eigenvalues $\lambda_1 = 1$, since when the states agree, they do not move anymore and $\lambda_2 = k_0^2 + \frac{N-1}{N}\sum_{i=1}^{\nu} k_i^2$ related to the convergence rate, which was computed before. The time evolution of $w(t)$ and $s(t)$ is thus given by

$$\begin{pmatrix} w(t) \\ s(t) \end{pmatrix} = c_1 \lambda_1^t a_1 + c_2 \lambda_2^t a_2$$

where c_1, c_2 are constants and a_1, a_2 are the eigenvectors associated to λ_1 and λ_2. At steady state the vector $(w(\infty), s(\infty))^T$ is aligned to the dominant eigenvector of Σ and thus $w(\infty) \approx c_1$. Simple calculations yield

$$w(\infty) = \alpha \frac{1}{N} x(0)^T \left(I - \Omega \right) x(0) \,,$$

where

$$\alpha = \frac{\sum_{i=1}^{\nu} k_i^2}{(1-N)\sum_{1=1}^{\nu} k_i^2 - k_0 N (k_0+2)}.$$

■

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